NLL Ultrasoft Running of the NRQCD potentials

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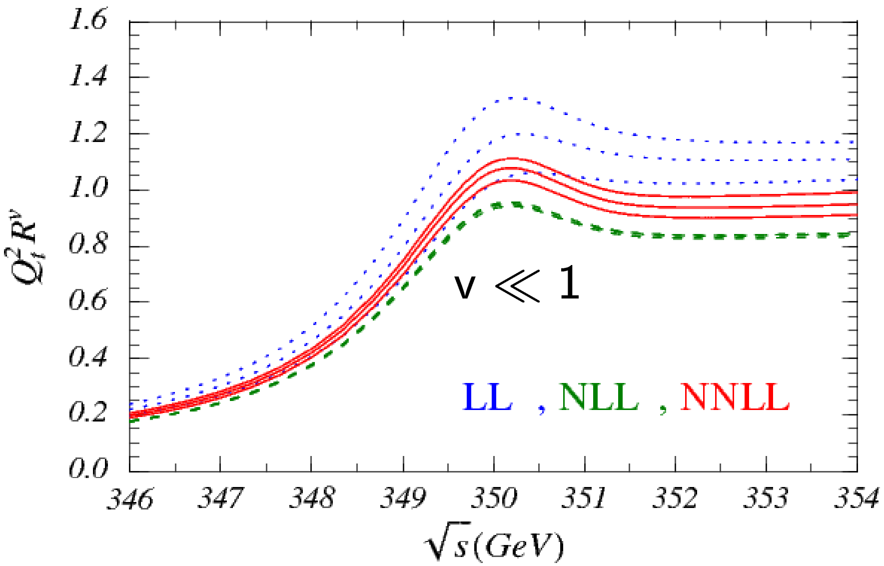
Outline

- Top Quark Threshold Physics
- Theoretical Issues
- vNRQCD
- Renormalization of the Potentials
- Status of Calculations
- Summary

Top Quark Threshold Physics

Top Physics at the ILC:

Focus: $t\bar{t}$ - production at threshold $(e^+ e^- \to t\bar{t})$



$\Gamma_t \approx 1.5\,\text{GeV} \gg \Lambda_{QCD} \quad \Rightarrow$

- non-pert. effects suppressed
- no sharp resonance peak

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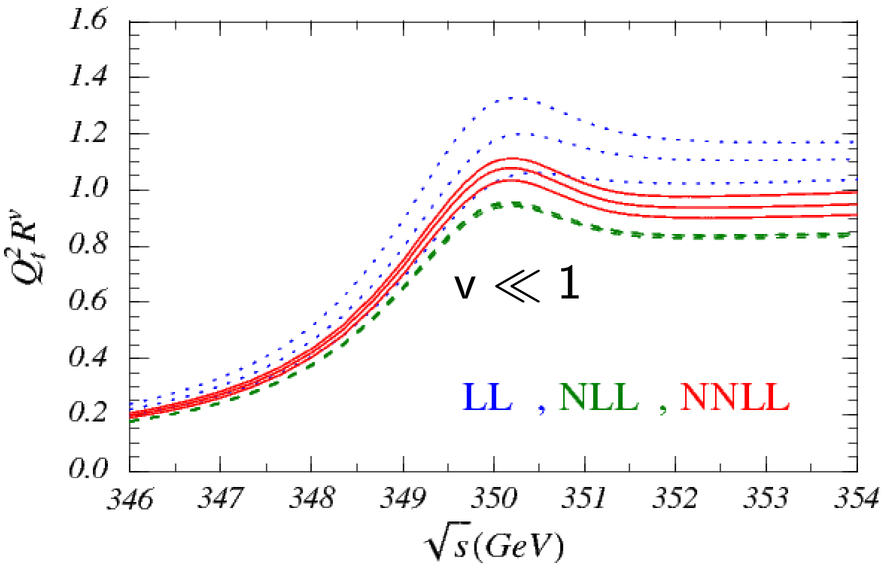

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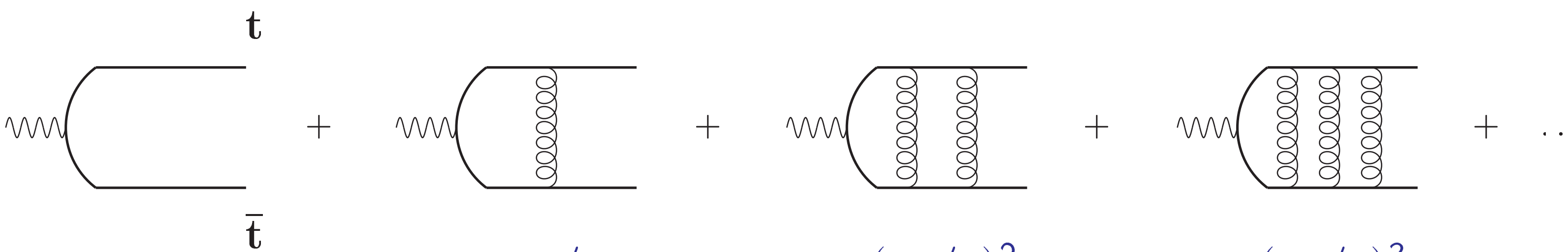
Aim: precise determination of

m_t	status: $\delta m_t \sim 100\,\text{MeV}$ ✓
$y_t,\ \alpha_s,\ \Gamma_t$	status: $\delta\sigma_{tot}^{theo}/\sigma_{tot} \sim 6\,\%$ (NNLL incomplete) needed: $< 3\,\%$

Theoretical Issues

Problem of Coulomb singularities:

Production threshold $\Rightarrow$ $v \sim \alpha_s \sim 0.1$ $\Rightarrow$ breakdown of perturbation theory



~ 1 $\quad \sim \alpha_s/v$ $\quad \sim (\alpha_s/v)^2$ $\quad \sim (\alpha_s/v)^3$

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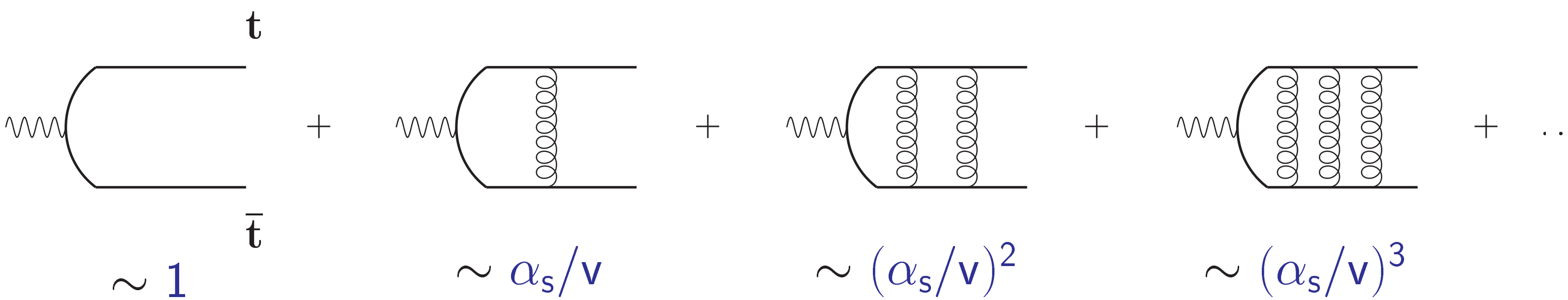

Solution:

non-rel. effective field theory NRQCD

$\rightarrow$ summation of $\left(\frac{\alpha_s}{v}\right)^n$- terms using a Schrödinger equation.

Theoretical Issues

Problem of large logarithms:

3 scales: $m_t \gg \vec{p} \sim m_t v \gg E_{kin} \sim m_t v^2 \quad (\sim \Gamma_t \gg \Lambda_{QCD})$

(soft) (ultrasoft)

$\Rightarrow$ log's at the dyn. scales:

$$\ln\left(\frac{m^2}{E^2}\right), \ \ln\left(\frac{m^2}{\mathbf{p}^2}\right), \ \ln\left(\frac{\mathbf{p}^2}{E^2}\right) \qquad \text{e.g.:} \quad \alpha_s \ln\left(\frac{m^2}{E^2}\right) \sim 1$$

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Solution:

two renormalization scales $\mu_s \sim mv,\ \mu_u \sim mv^2$

$E \sim \frac{\mathbf{p}^2}{m} \to$ correlation: $\mu_u \sim \frac{\mu_s^2}{m} \to \mu_s = m\nu,\ \mu_u = m\nu^2 \quad \Rightarrow$ "v"NRQCD

$1 \geq \nu \geq 0$

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$1 \geq \nu \geq 0$

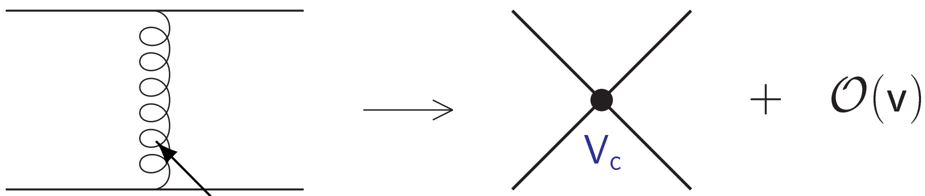
$\Rightarrow$ RGE's sum up $[\alpha_s \ln v]^n, \ \alpha_s [\alpha_s \ln v]^n, \ \alpha_s^2 [\alpha_s \ln v]^n, \ \ldots$ terms.

LL NLL NNLL

vNRQCD

Construction of NRQCD:

Integrate out non-resonant degrees of freedom, e.g.



"potential" Gluon: $k^\mu \sim (mv^2, mv)$

$$\Rightarrow \; \mathcal{L}_{NR} = \mathcal{L}_{\rm kin} + \Big[\frac{\mathcal{V}_c}{\mathbf{k}^2} + \frac{\mathcal{V}_k \pi^2}{m\mathbf{k}} + \frac{\mathcal{V}_r(\mathbf{p}^2+\mathbf{p}'^2)}{2m^2\mathbf{k}^2} + \frac{\mathcal{V}_2}{m^2} + \frac{\mathcal{V}_s}{m^2}\mathbf{S}^2 + \ldots \Big] \psi^\dagger\psi\,\chi^\dagger\chi + \ldots$$

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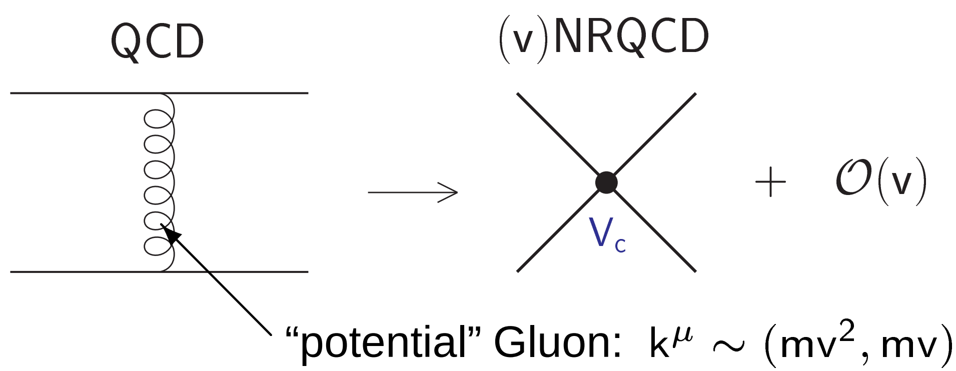

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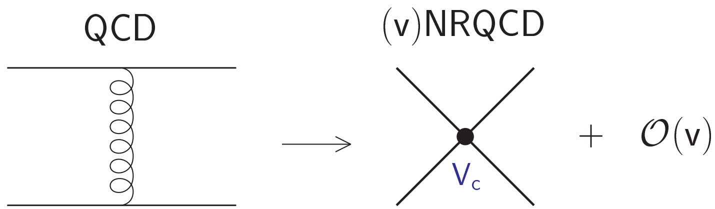
Separate center-of-mass motion

$\Rightarrow$ Schrödinger Eq.: $\quad E\,\Psi = \Big[\, \frac{\mathbf{k}^2}{m} + V + \ldots \Big]\, \Psi$

Green fct.: $\Big[-\frac{\nabla_{\vec r}^2}{m} + V(r) - E \Big]\, G(\vec r, \vec r\,', E) = \delta^{(3)}(\vec r - \vec r\,')$, $\quad \boxed{\sigma_{\rm tot} \propto {\rm Im}[G(0,0,E)]}$

vNRQCD

- Non-resonant dof's integrated out, e.g.:



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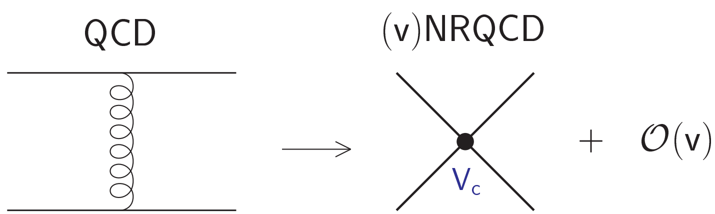
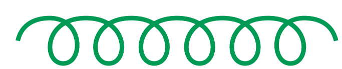


- Resonant dof's $\rightarrow$ fields in the eff. Lagrangian:

non-rel. quark:	$(E, \mathbf{p}) \sim (mv^2, mv)$	$\psi_{\mathbf{p}}(x)$	
soft gluon:	$(q_0, \mathbf{q}) \sim (mv, mv)$	$A_{\mathbf{q}}(x)$	
ultrasoft gluon:	$(q_0, \mathbf{q}) \sim (mv^2, mv^2)$	$A(x)$	

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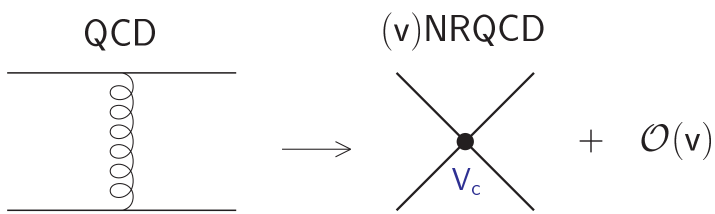

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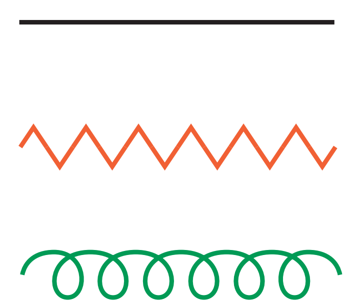


- Systematic expansion in $v \rightarrow$ consistent power counting in $v \sim \alpha_s$

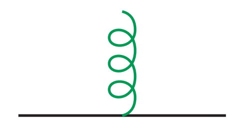
vNRQCD

[Luke, Manohar, Rothstein]

$\underline{\mathcal{L}_{\mathrm{vNRQCD}} = \mathcal{L}_{\mathrm{usoft}} + \mathcal{L}_{\mathrm{pot}} + \mathcal{L}_{\mathrm{soft}}}$

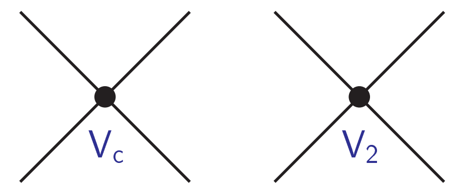
$D^\mu = \partial^\mu + igA^\mu(x)$

$$\mathcal{L}_{\mathrm{usoft}} : \quad \psi^\dagger_{\mathbf{p}}(x) \left\{ iD^0 - \frac{(\mathbf{p} - i\mathbf{D})^2}{2m} + \ldots \right\} \psi_{\mathbf{p}}(x) + \ldots$$



$$\mathcal{L}_{\mathrm{pot}} : \quad -V\, \psi^\dagger_{\mathbf{p}'} \psi_{\mathbf{p}} \chi^\dagger_{-\mathbf{p}'} \chi_{-\mathbf{p}} + \ldots \qquad (\mathbf{k} = \mathbf{p}' - \mathbf{p})$$

$$\boxed{V \sim \frac{\mathcal{V}_c}{\mathbf{k}^2} + \frac{\mathcal{V}_k \pi^2}{m\mathbf{k}} + \frac{\mathcal{V}_r(\mathbf{p}^2 + \mathbf{p}'^2)}{2m^2\mathbf{k}^2} + \frac{\mathcal{V}_2}{m^2} + \frac{\mathcal{V}_s}{m^2}\mathbf{S}^2 + \ldots}$$



vNRQCD

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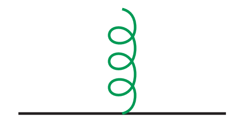

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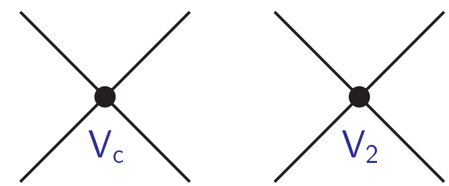
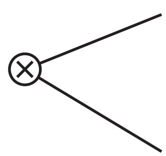


external production/annihilation current:

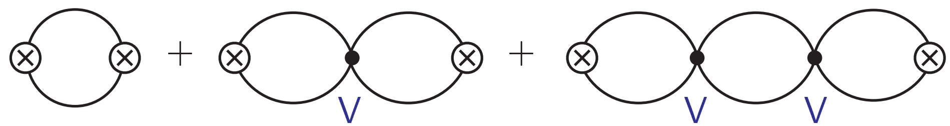


$$\sim c_1(\mu)\cdot \underbrace{\vec{j}_1^{\,\rm eff}(x)}_{\psi^\dagger_{\mathbf{p}}\vec{\sigma}(i\sigma_2)\chi^*_{-\mathbf{p}}} + \ldots \qquad \text{(CMS)}$$

Renormalization of the Potentials

Anomalous dimension of V contributes to $\sigma(\mathrm{e^+ e^-} \to \mathrm{t\bar{t}})$:

$\sigma_{\rm tot} \sim \mathrm{Im}\Big[$



$+ \cdots \Big]$

$$\sim |c_1(\mu)|^2 \cdot \mathrm{Im}\left[-i \int d^4x \; e^{i\hat{q}x} \langle 0 | \, T \, \vec{j}_1^{\,\rm eff\,*}(x) \, \vec{j}_1^{\,\rm eff}(0) \, | 0 \rangle \right]$$

$$\sim |c_1(\mu)|^2 \cdot \mathrm{Im}\left[G(0,0,E,\mu)\right]$$

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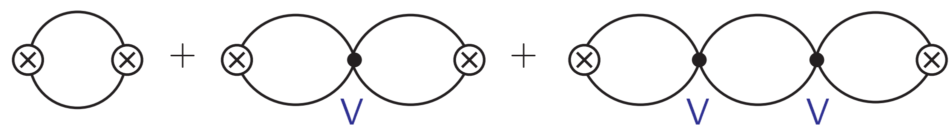

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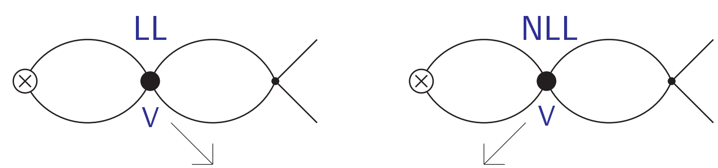
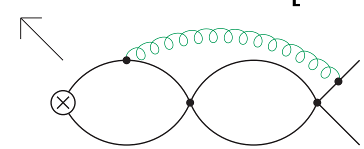
current renormalization



$$\ln\left[\frac{c_1(\mu)}{c_1(m)}\right] = \underbrace{\xi^{\rm LL}}_{0} + \xi^{\rm NLL} + \xi^{\rm NNLL}_{\rm mix} + \xi^{\rm NNLL}_{\rm non-mix}$$

[Hoang]



[Luke, Manohar, Rothstein, Pineda, Hoang, Stewart]

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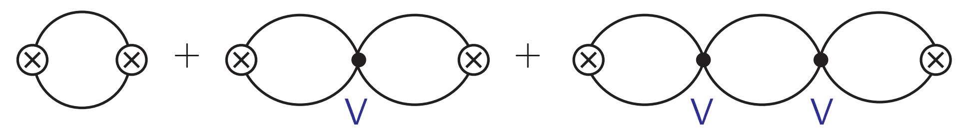

$$\Big]$$

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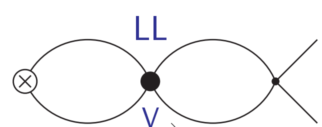

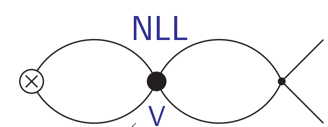

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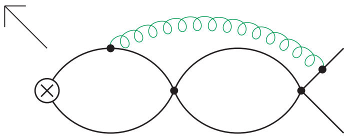

[Luke, Manohar, Rothstein, Pineda, Hoang, Stewart]

missing $\Rightarrow$ $V^{\rm NLL}(\mu)$ needed!

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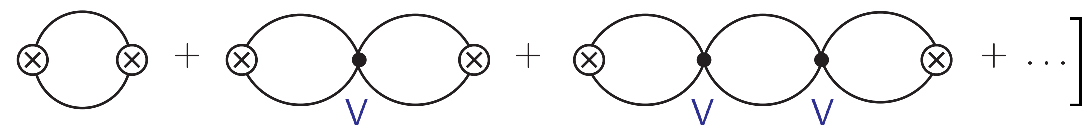
Anomalous dimension of V contributes to $\sigma(e^+ e^- \to t\,\bar{t})$:

$$\sigma_{\rm tot} \sim \mathrm{Im}\Big[\quad\Big]$$



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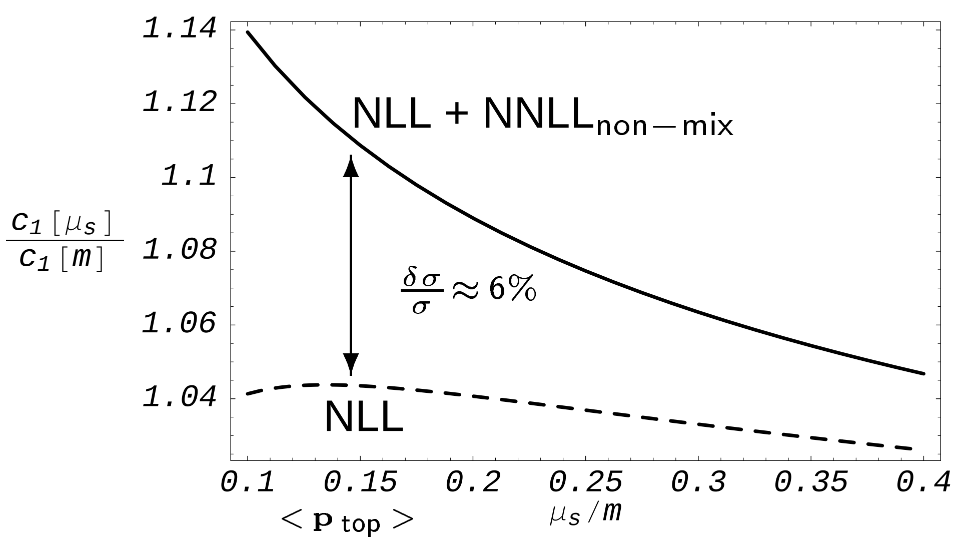


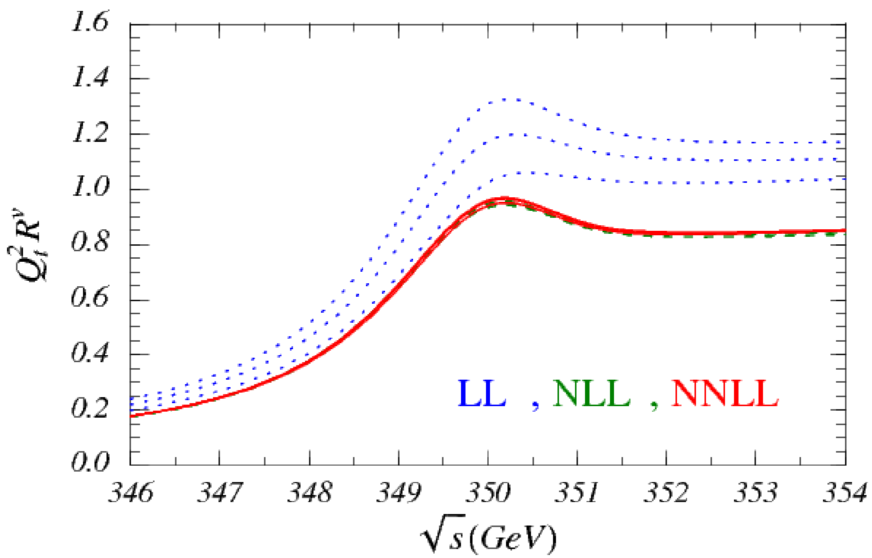
missing $\mathrm{NNLL}_{\rm mix}$ contribution
may reduce theortical error of $\sigma_{\rm tot}$

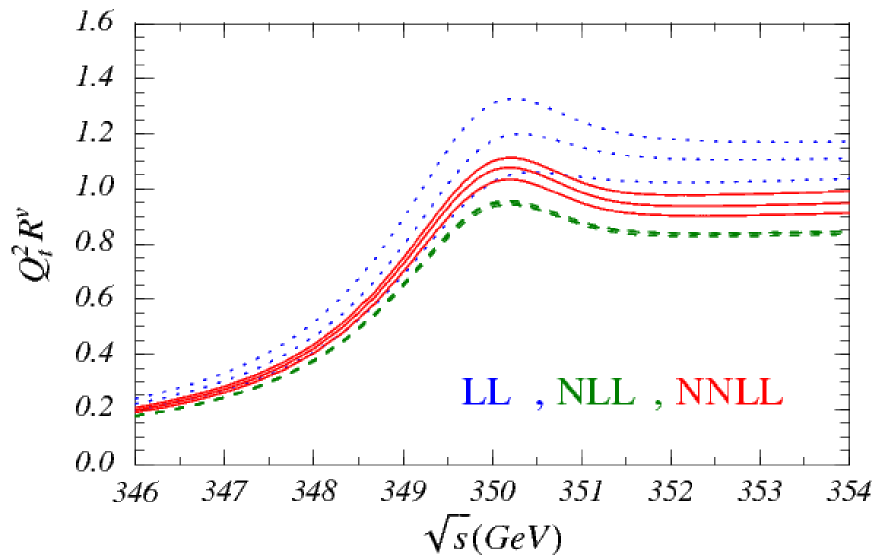
$\Rightarrow$ $\boxed{V^{\rm NLL}(\mu)}$ needed!

Renormalization of the Potentials

$\text{NNLL}_{\text{nonmix}}$ contribution to c_1 has large effect on σ_{tot}:

[Hoang]





NLL running of c_1 is used for NNLL

$\Rightarrow$ small scale (ν) dependence

$\text{NNLL}_{\text{nonmix}}$ of c_1 is included in NNLL

$\Rightarrow$ large scale (ν) dependence

$\Rightarrow$ large theoretical uncertainty ($\sim 6\,\%$) of σ_{tot}

Renormalization of the Potentials

NLL anomalous dimension of $\mathsf{c}_1(\nu)$:

$$\mu\frac{\partial}{\partial\nu}\ln[\mathsf{c}_1(\nu)] \;=\; -\frac{\mathcal{V}_{\mathsf{c}}(\nu)}{16\pi^2}\left[\frac{\mathcal{V}_{\mathsf{c}}(\nu)}{4}+\mathcal{V}_2(\nu)+\mathcal{V}_{\mathsf{r}}(\nu)+\mathbf{S}^2\,\mathcal{V}_{\mathsf{s}}(\nu)\right]+\frac{1}{2}\mathcal{V}_{\mathsf{k}}(\nu)$$

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NLL anomalous dimension of the potentials:

- soft NLL running of V_s known $\rightarrow$ tiny effect [Pineda, Steinhauser, Penin, Smirnov]

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  at threshold: $\boxed{\alpha_s(mv^2) \simeq 0.27 \;>\; \alpha_s(mv) \simeq 0.15}$ $(v \simeq 0.1)$

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- Relevant potentials for $\mathsf{c}_1(\nu)$ at NNLL$_{\mathsf{mix}}$, which are affected by NLL ultrasoft
  renormalization: $\boxed{\frac{\mathcal{V}_\mathsf{k}\pi^2}{\mathsf{m}\mathbf{k}}\,,\;\; \frac{\mathcal{V}_\mathsf{r}(\mathbf{p}^2+\mathbf{p}'^2)}{2\mathsf{m}^2\mathbf{k}^2}\,,\;\; \frac{\mathcal{V}_2}{\mathsf{m}^2}}$

Renormalization of the Potentials

NLL anomalous dimension of $c_1(\nu)$:

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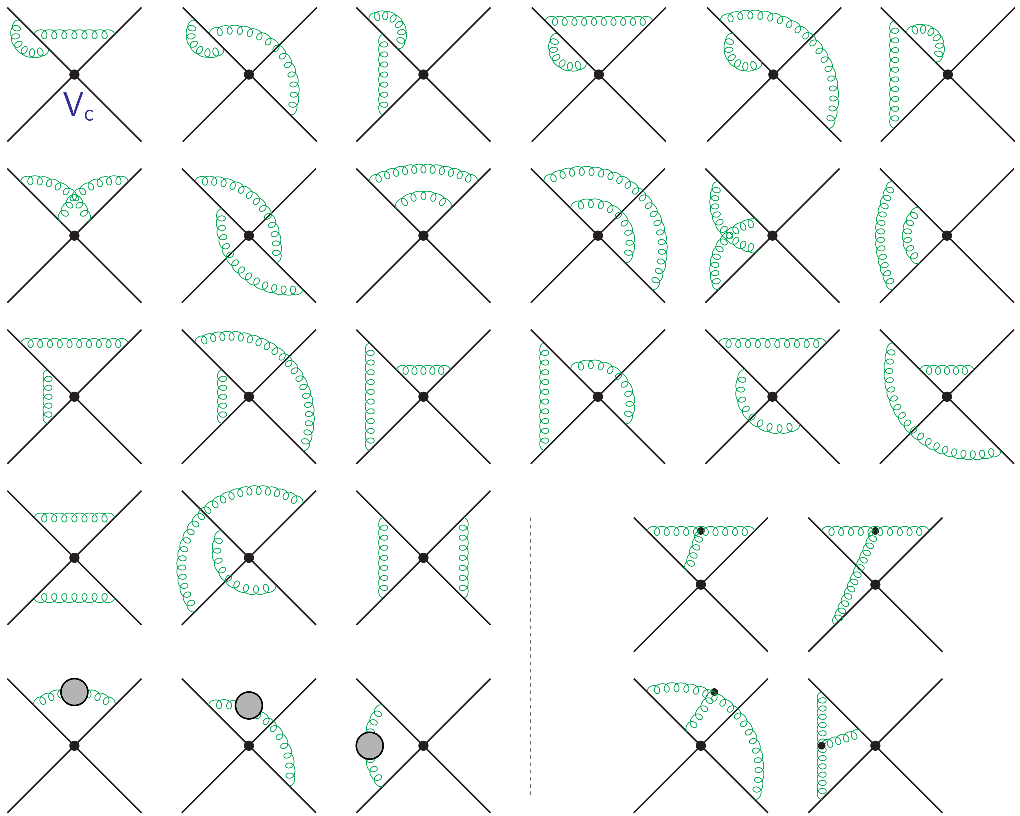
- soft NLL running of V_s known $\rightarrow$ tiny effect [Pineda, Steinhauser, Penin, Smirnov]
- Ultrasoft contributions supposed to be dominant, since at threshold: $\alpha_s(mv^2) \simeq 0.27 \;>\; \alpha_s(mv) \simeq 0.15$ $(v \simeq 0.1)$
- Relevant potentials for $c_1(\nu)$ at $NNLL_{mix}$, which are affected by NLL ultrasoft renormalization: $\frac{\mathcal{V}_k\pi^2}{m\mathbf{k}}\,,\;\frac{\mathcal{V}_r(\mathbf{p}^2+\mathbf{p}'^2)}{2m^2\mathbf{k}^2}\,,\;\frac{\mathcal{V}_2}{m^2}$
- Cross check: ultrasoft $\mathcal{O}(v^0)$-corrections to $V = 0$ ✓
  follows from the form of the non. rel. Lagragian
  (field redefinition: $\psi_{\mathbf{p}}^\dagger iD^0\psi_{\mathbf{p}} \rightarrow \Phi_{\mathbf{p}}^\dagger i\partial^0\Phi_{\mathbf{p}}$, $\psi_{\mathbf{p}}(x) = W(x)\,\Phi_{\mathbf{p}}(x)$)

Renormalization of the Potentials

- Some 2-loop diagrams contributing to the NLL running of V_2 and V_r:



e.g.:

$$\Rightarrow \delta V_2^{2\,\mathrm{loop}} \xrightarrow{\mathrm{RGE}} V_2^{\mathrm{NLL}}(\mu)$$

V_2^{NLL} and V_r^{NLL} complete ✓

[Stahlhofen, Hoang] to appear

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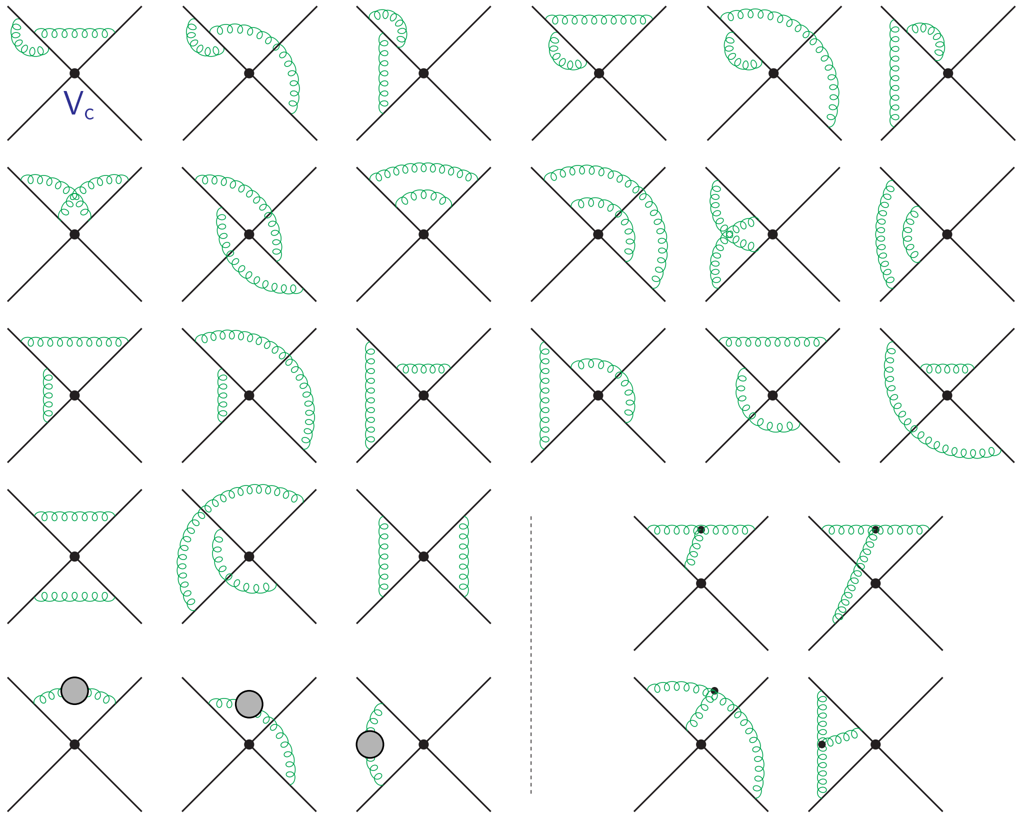

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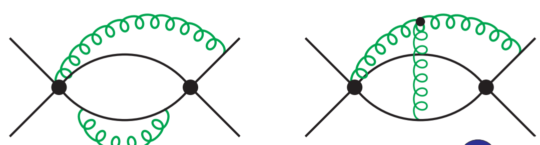
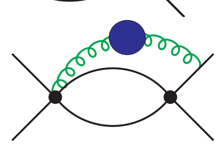
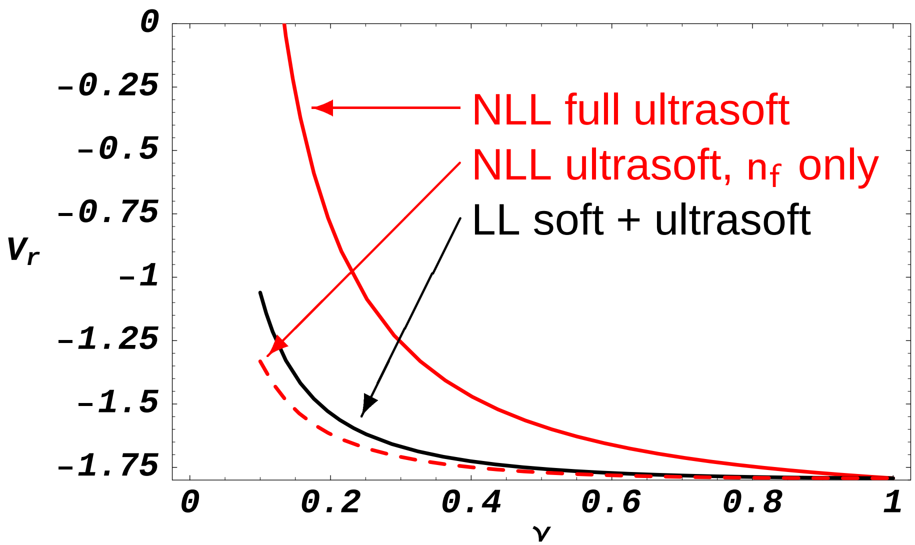
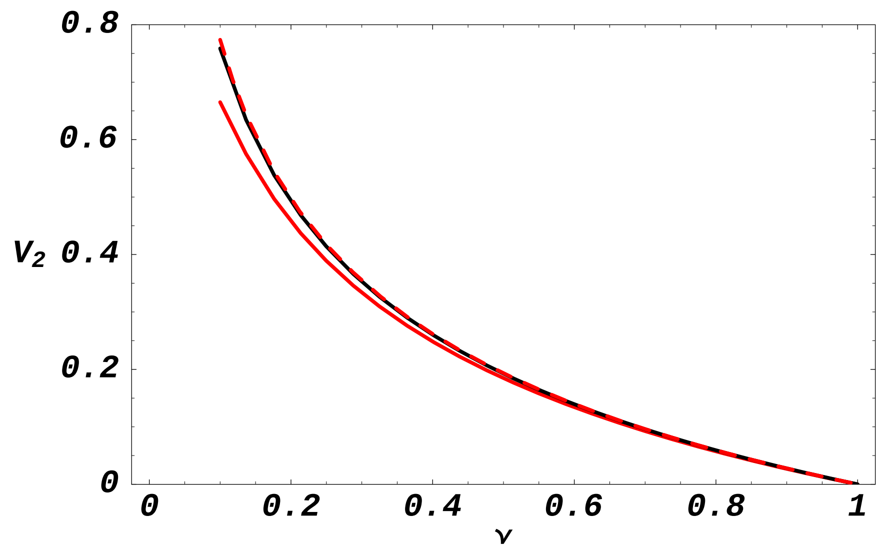
- To V_k diagrams like  contribute: work in progress!

(fermionic (n_f) contributions e.g. from  complete ✓)

Status of Calculations

Preliminary results for $\frac{1}{m^2}$ potentials:





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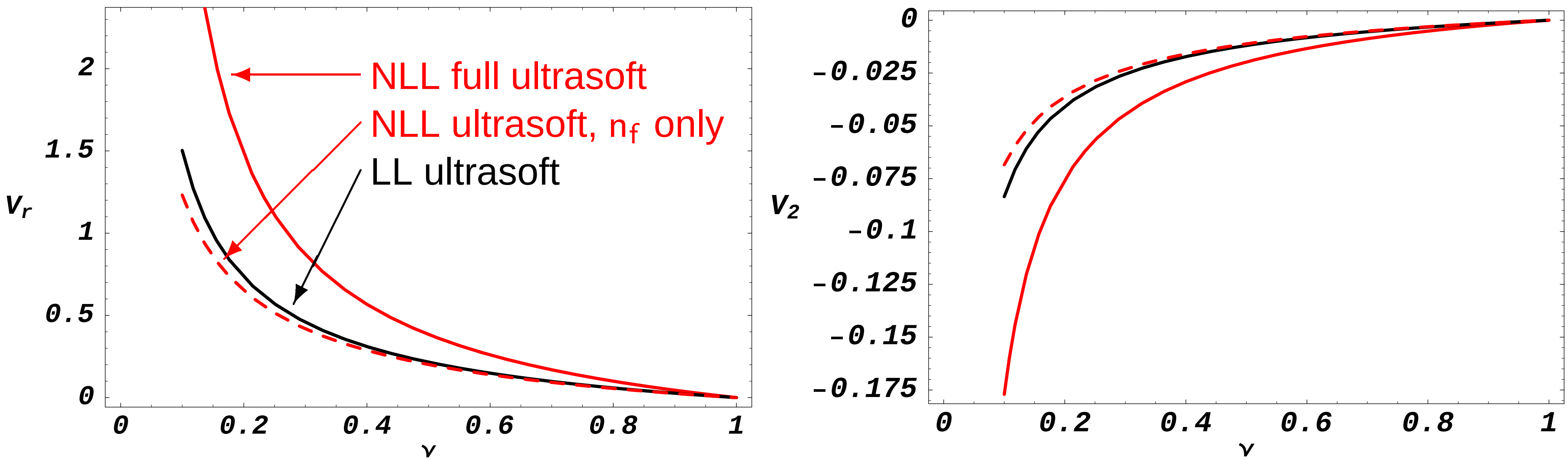

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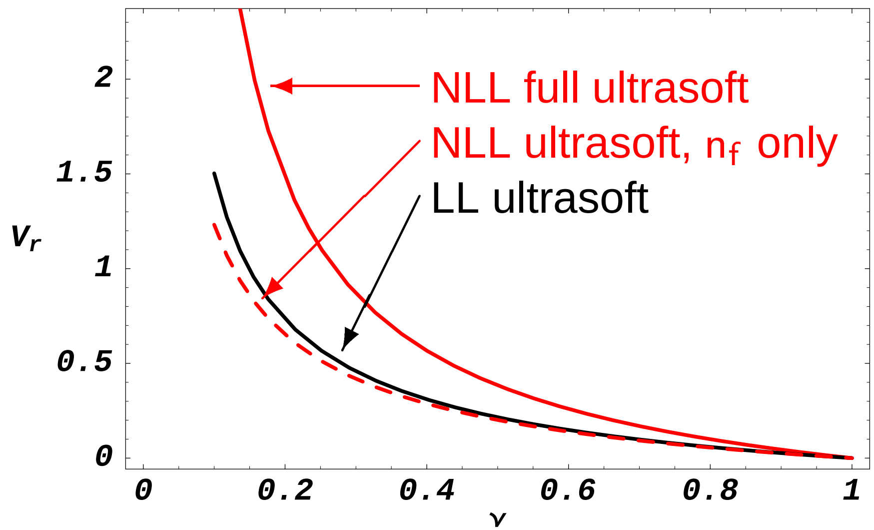
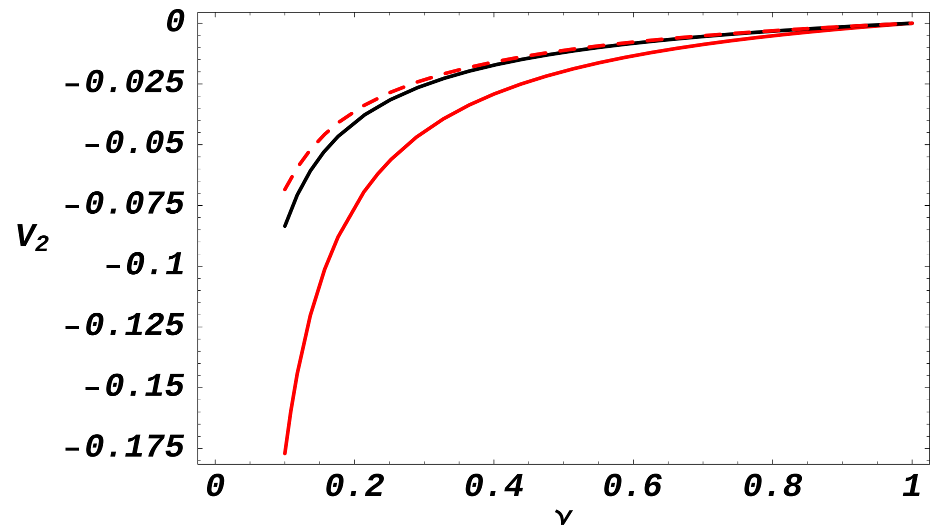
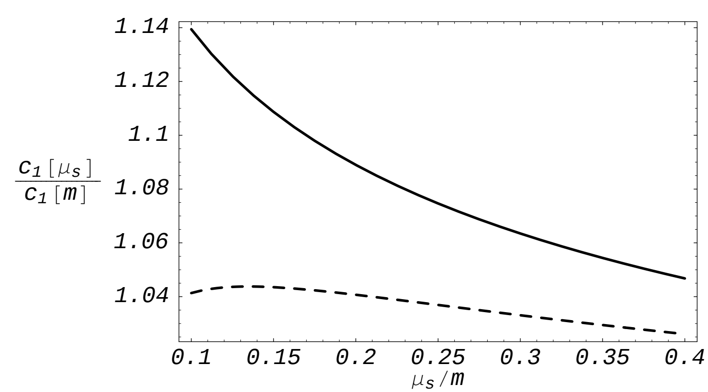




- Analysis shows that **usoft LL ∼ usoft NLL**

⇒ Big $\mathrm{NNLL_{mix}}$ contributions to c_1 expected

⇒ may compensate $\mathrm{NNLL_{nonmix}}$ and reduce ν dependence of c_1!



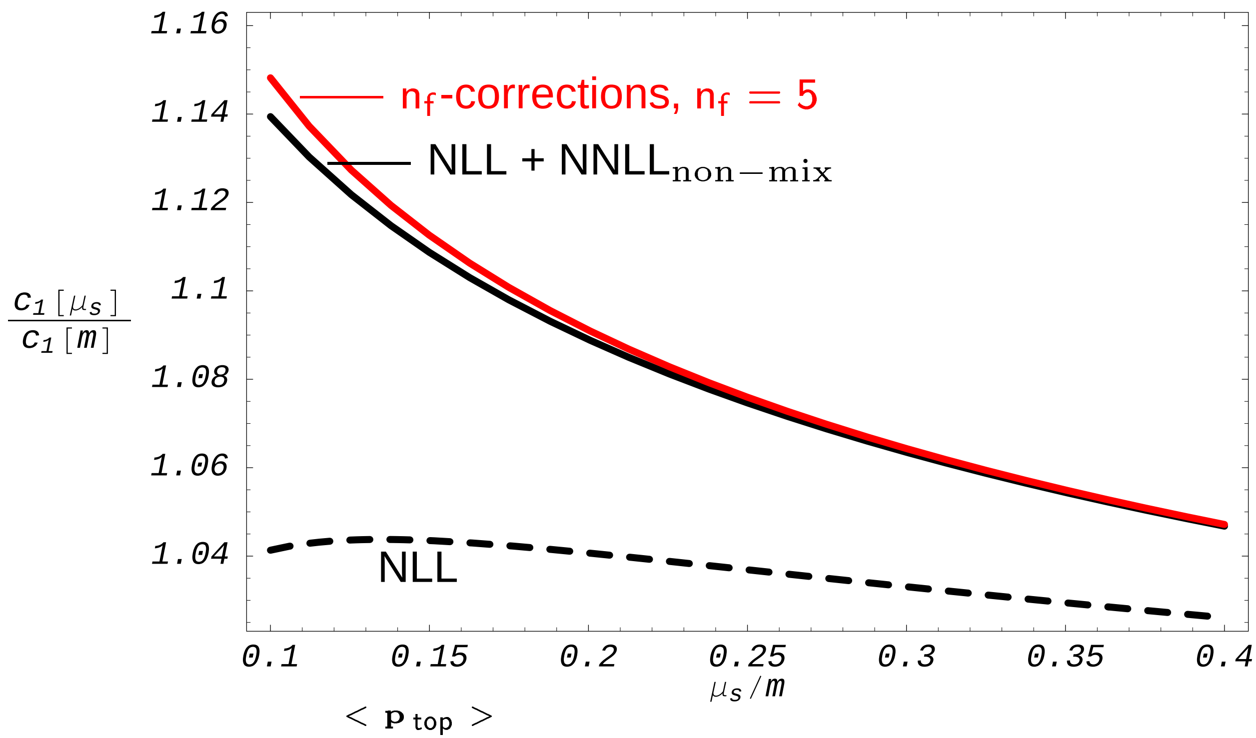
Summary

Ultrasoft NLL running of the potentials V_k, V_r, V_2 is essential for a precise prediction of $\sigma_{tot}(e^+ e^- \to t\,\bar{t})$ at threshold.

Current status of the calculation:

Contribution	V_k	V_r	V_2	V_s
soft + usoft LL	✓	✓	✓	✓
usoft NLL n_f	✓	✓	✓	0
full usoft NLL	–	✓	✓	0
soft NLL	–	–	–	✓

Old Result



1.16
1.14
1.12
1.1
1.08
1.06
1.04
$\frac{c_1[\mu_s]}{c_1[m]}$
n_f-corrections, $n_f = 5$
NLL + $\text{NNLL}_{\text{non-mix}}$
NLL
0.1
0.15
0.2
0.25
0.3
0.35
0.4
μ_s / m
$< \mathbf{p}_{\text{top}} >$

Extra Formulae

$$\nu \frac{\partial}{\partial \nu} \ln[c_1(\nu)] = -\frac{\mathcal{V}_c^{(0)}(\nu)}{16\pi^2}\left[\frac{\mathcal{V}_c^{(0)}(\nu)}{4} + \mathcal{V}_2^{(2)}(\nu) + \mathcal{V}_r^{(2)}(\nu) + \mathbf{S}^2\, \mathcal{V}_s^{(2)}(\nu)\right.$$
$$\left. + \frac{1}{2}\mathcal{V}_k^{(1)}(\nu) + \alpha_s^2(m\nu)\left[3\mathcal{V}_{k1}^{(1)}(\nu) + 2\mathcal{V}_{k2}^{(1)}(\nu)\right]\right]$$

$$v \cong \alpha_s(mv) = \frac{4\pi}{\beta_0 \ln(m^2 v^2/\Lambda_{QCD}^2)} \quad \Rightarrow \quad v \cong \alpha_s \cong 0.14$$

$$v \equiv \sqrt{\frac{\sqrt{s}-2m_t}{m_t}} \rightarrow \sqrt{\frac{\sqrt{s}-2m_t+i\Gamma_t}{m_t}}$$ [Fadin, Khoze]