

NLL Ultrasoft Running of the NRQCD potentials

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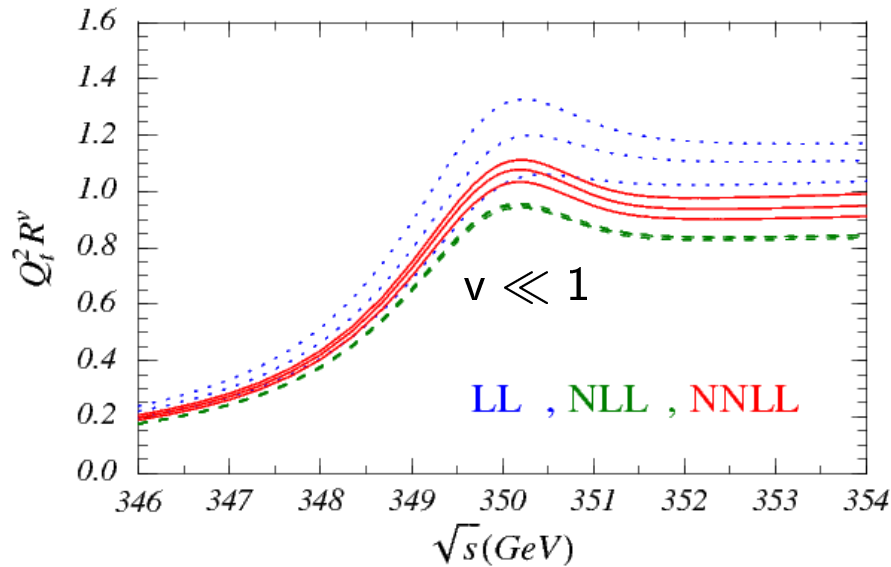
Outline

- Top Quark Threshold Physics
- Theoretical Issues
- v NRQCD
- Renormalization of the Potentials
- Status of Calculations
- Summary

Top Quark Threshold Physics

Top Physics at the ILC:

Focus: $t\bar{t}$ - production at threshold ($e^+e^- \rightarrow t\bar{t}$)



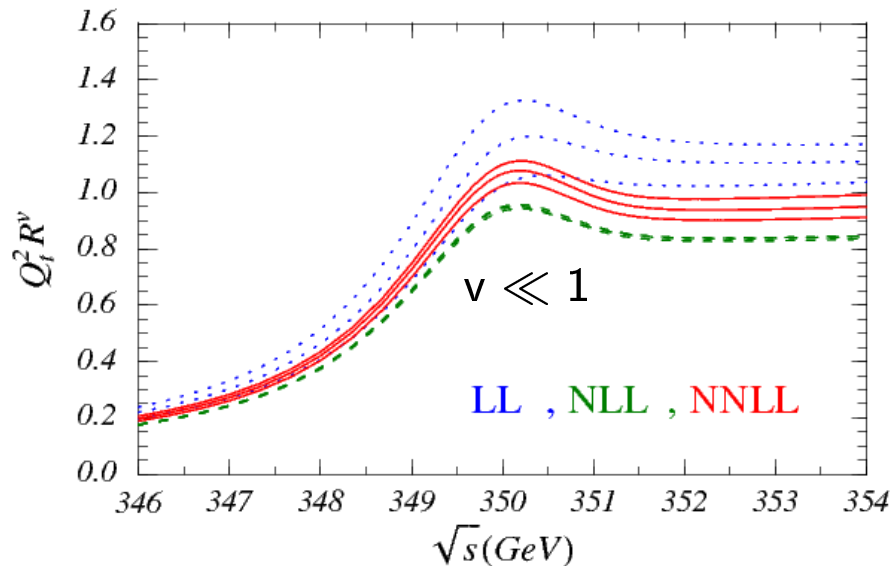
$$\Gamma_t \approx 1.5 \text{ GeV} \gg \Lambda_{\text{QCD}} \Rightarrow$$

- non-pert. effects suppressed
- no sharp resonance peak

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Aim: precise determination of

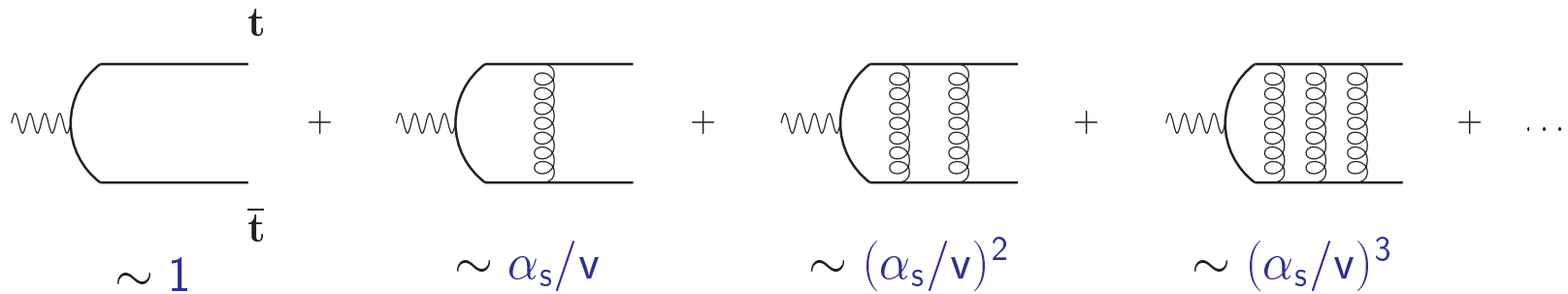
m_t status: $\delta m_t \sim 100 \text{ MeV} \checkmark$

y_t, α_s, Γ_t status: $\delta \sigma_{\text{tot}}^{\text{theo}} / \sigma_{\text{tot}} \sim 6\%$ (NNLL incomplete) **needed: $< 3\%$**

Theoretical Issues

Problem of Coulomb singularities:

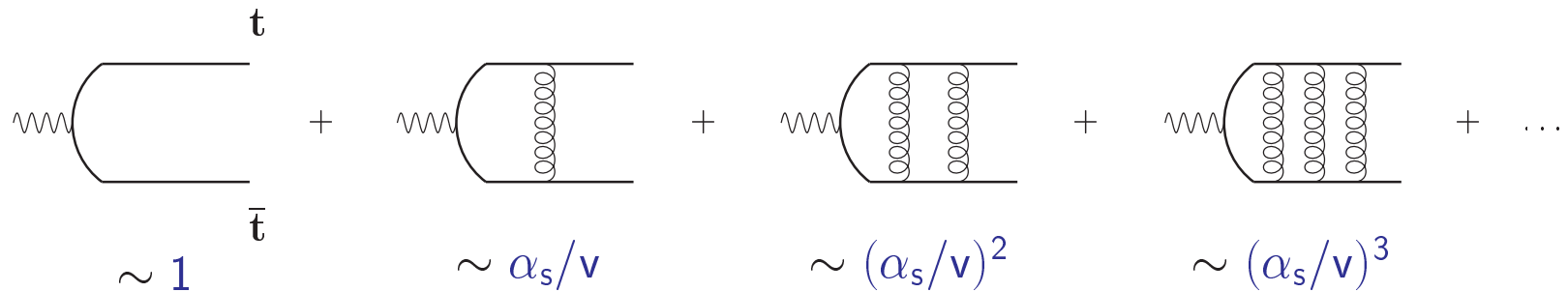
Production threshold $\Rightarrow v \sim \alpha_s \sim 0.1 \Rightarrow$ breakdown of perturbation theory



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The diagram shows a series of Feynman diagrams for top quark production. Each diagram consists of a wavy line on the left representing a gluon, which splits into a top quark (t) and an anti-top quark (t-bar). The top quark and anti-top quark lines then interact via a series of gluon exchanges (represented by vertical wavy lines) before exiting to the right. The diagrams are summed together, with the first term being the tree-level process and subsequent terms representing higher-order corrections. The terms are labeled with their respective orders in the coupling constant α_s/v .

$$\sim 1 + \sim \alpha_s/v + \sim (\alpha_s/v)^2 + \sim (\alpha_s/v)^3 + \dots$$

Solution:

non-rel. effective field theory NRQCD

\rightarrow summation of $\left(\frac{\alpha_s}{v}\right)^n$ - terms using a Schrödinger equation.

Theoretical Issues

Problem of large logarithms:

$$\boxed{\text{3 scales: } m_t \gg \vec{p} \sim m_t v \gg E_{\text{kin}} \sim m_t v^2 \quad (\sim \Gamma_t \gg \Lambda_{\text{QCD}})}$$

(soft) (ultrasoft)

⇒ log's at the dyn. scales:

$$\ln\left(\frac{m^2}{E^2}\right), \ln\left(\frac{m^2}{p^2}\right), \ln\left(\frac{p^2}{E^2}\right) \quad \text{e.g.: } \alpha_s \ln\left(\frac{m^2}{E^2}\right) \sim 1$$

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Solution:

two renormalization scales $\mu_s \sim m v, \mu_u \sim m v^2$

$$E \sim \frac{p^2}{m} \rightarrow \text{correlation: } \mu_u \sim \frac{\mu_s^2}{m} \rightarrow \mu_s = m \nu, \mu_u = m \nu^2 \Rightarrow \boxed{\text{“v”NRQCD}}$$

$1 \geq \nu \geq 0$

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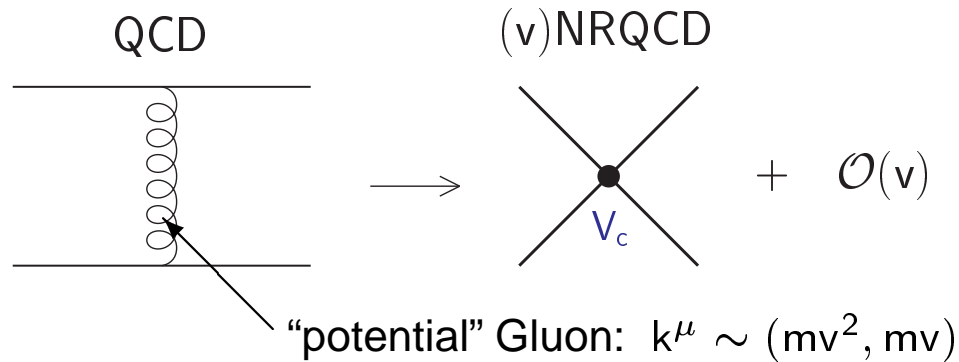
⇒ RGE's sum up $[\alpha_s \ln v]^n$, $\alpha_s [\alpha_s \ln v]^n$, $\alpha_s^2 [\alpha_s \ln v]^n$, ... terms.

LL NLL NNLL

v NRQCD

Construction of NRQCD:

Integrate out non-resonant degrees of freedom, e.g.

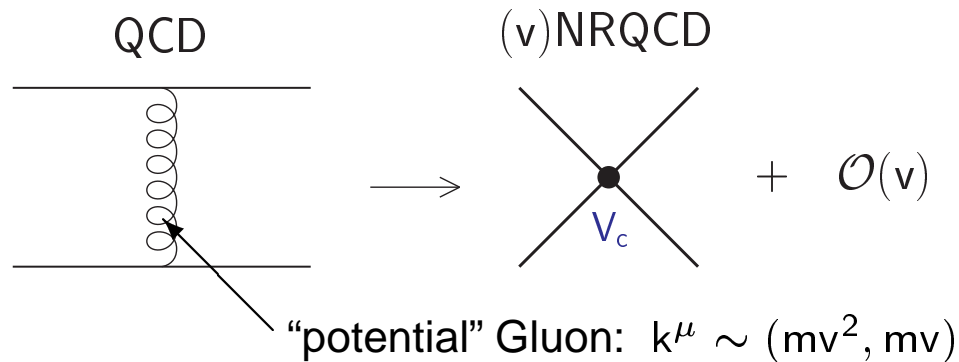


$$\Rightarrow \mathcal{L}_{\text{NR}} = \mathcal{L}_{\text{kin}} + \left[\frac{\mathcal{V}_c}{\mathbf{k}^2} + \frac{\mathcal{V}_k \pi^2}{m\mathbf{k}} + \frac{\mathcal{V}_r (\mathbf{p}^2 + \mathbf{p}'^2)}{2m^2 \mathbf{k}^2} + \frac{\mathcal{V}_2}{m^2} + \frac{\mathcal{V}_s}{m^2} \mathbf{S}^2 + \dots \right] \psi^\dagger \psi \chi^\dagger \chi + \dots$$

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Separate center-of-mass motion

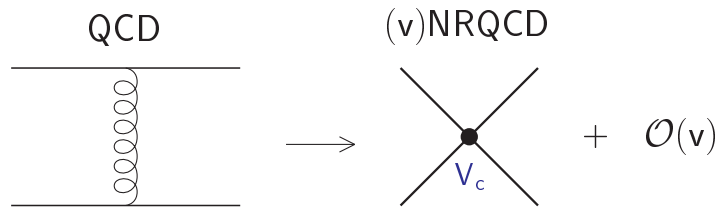
$$\Rightarrow \text{Schrödinger Eq.: } E \Psi = \left[\frac{\mathbf{k}^2}{m} + \mathbf{V} + \dots \right] \Psi$$

$$\text{Green fct.: } \left[-\frac{\nabla_{\vec{r}}^2}{m} + \mathbf{V}(\mathbf{r}) - E \right] \mathbf{G}(\vec{r}, \vec{r}', E) = \delta^{(3)}(\vec{r} - \vec{r}'),$$

$$\sigma_{\text{tot}} \propto \text{Im}[\mathbf{G}(0, 0, E)]$$

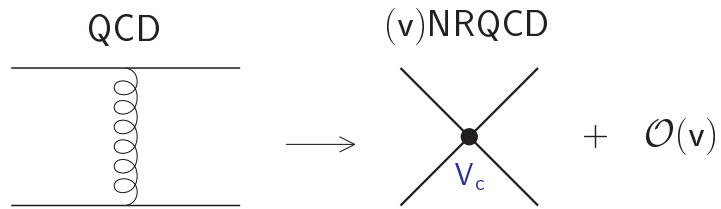
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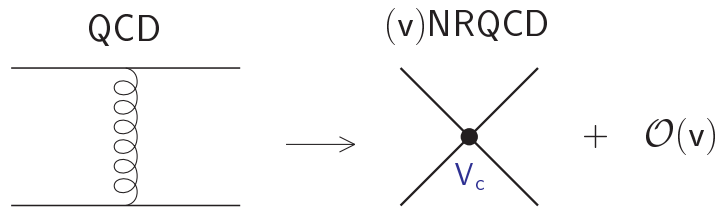


- Resonant dof's \rightarrow fields in the eff. Lagrangian:

non-rel. quark:	$(E, \mathbf{p}) \sim (mv^2, mv)$	$\psi_{\mathbf{p}}(x)$	
soft gluon:	$(q_0, \mathbf{q}) \sim (mv, mv)$	$A_{\mathbf{q}}(x)$	
ultrasoft gluon:	$(q_0, \mathbf{q}) \sim (mv^2, mv^2)$	$A(x)$	

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- Systematic expansion in $v \rightarrow$ consistent power counting in $v \sim \alpha_s$

vNRQCD

[Luke, Manohar, Rothstein]

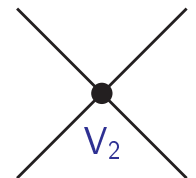
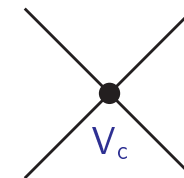
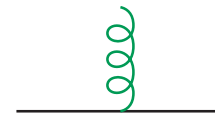
$$\underline{\mathcal{L}_{\text{vNRQCD}} = \mathcal{L}_{\text{usoft}} + \mathcal{L}_{\text{pot}} + \mathcal{L}_{\text{soft}}}$$

$$\mathcal{L}_{\text{usoft}} : \psi_{\mathbf{p}}^\dagger(x) \left\{ iD^0 - \frac{(\mathbf{p} - i\mathbf{D})^2}{2m} + \dots \right\} \psi_{\mathbf{p}}(x) + \dots$$

$$\mathcal{L}_{\text{pot}} : -V \psi_{\mathbf{p}}^\dagger \psi_{\mathbf{p}} \chi_{-\mathbf{p}'}^\dagger \chi_{-\mathbf{p}} + \dots \quad (\mathbf{k} = \mathbf{p}' - \mathbf{p})$$

$$V \sim \frac{\mathcal{V}_c}{\mathbf{k}^2} + \frac{\mathcal{V}_k \pi^2}{m\mathbf{k}} + \frac{\mathcal{V}_r (\mathbf{p}^2 + \mathbf{p}'^2)}{2m^2 \mathbf{k}^2} + \frac{\mathcal{V}_2}{m^2} + \frac{\mathcal{V}_s}{m^2} \mathbf{S}^2 + \dots$$

$$D^\mu = \partial^\mu + igA^\mu(x)$$



vNRQCD

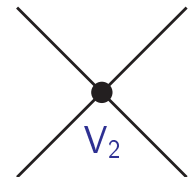
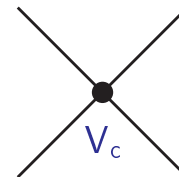
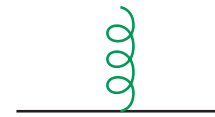
[Luke, Manohar, Rothstein]

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external production/annihilation current:

$$\begin{array}{c} \text{⊗} \\ \diagup \\ \diagdown \end{array} \sim c_1(\mu) \cdot \underbrace{\vec{j}_1^{\text{eff}}(x)}_{\psi_{\mathbf{p}}^\dagger \vec{\sigma}(i\sigma_2) \chi_{-\mathbf{p}}^*} + \dots \quad (\text{CMS})$$

Renormalization of the Potentials

Anomalous dimension of V contributes to $\sigma(e^+ e^- \rightarrow t \bar{t})$:

$$\begin{aligned}
 \sigma_{\text{tot}} &\sim \text{Im} \left[\text{Diagram 1} + \text{Diagram 2} + \text{Diagram 3} + \dots \right] \\
 &\sim |c_1(\mu)|^2 \cdot \text{Im} \left[-i \int d^4x e^{i\hat{q}x} \langle 0 | T \vec{j}_1^{\text{eff}*}(x) \vec{j}_1^{\text{eff}}(0) | 0 \rangle \right] \\
 &\sim |c_1(\mu)|^2 \cdot \text{Im} [G(0, 0, E, \mu)]
 \end{aligned}$$

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Diagram 1: A circle with two external lines, each marked with a cross (⊗).
 Diagram 2: Two circles connected at a vertex labeled V . Each external line is marked with a cross (⊗).
 Diagram 3: Three circles connected in a chain at two vertices labeled V . Each external line is marked with a cross (⊗).

$$\sim |c_1(\mu)|^2 \cdot \text{Im} \left[-i \int d^4x e^{i\hat{q}x} \langle 0 | T \vec{j}_1^{\text{eff}*}(x) \vec{j}_1^{\text{eff}}(0) | 0 \rangle \right]$$

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current renormalization

$$\ln \left[\frac{c_1(\mu)}{c_1(m)} \right] = \underbrace{\xi^{\text{LL}}}_0 + \xi^{\text{NLL}} + \xi_{\text{mix}}^{\text{NNLL}} + \xi_{\text{non-mix}}^{\text{NNLL}} \quad [\text{Hoang}]$$

Diagram 1 (LL): Two circles connected at a vertex labeled V .
 Diagram 2 (NLL): Two circles connected at a vertex labeled V , with a third line attached to the right circle.
 Diagram 3 (NNLL): Two circles connected at a vertex labeled V , with a third line attached to the right circle and a wavy green line connecting the two vertices.

[Luke, Manohar, Rothstein,
Pineda, Hoang, Stewart]

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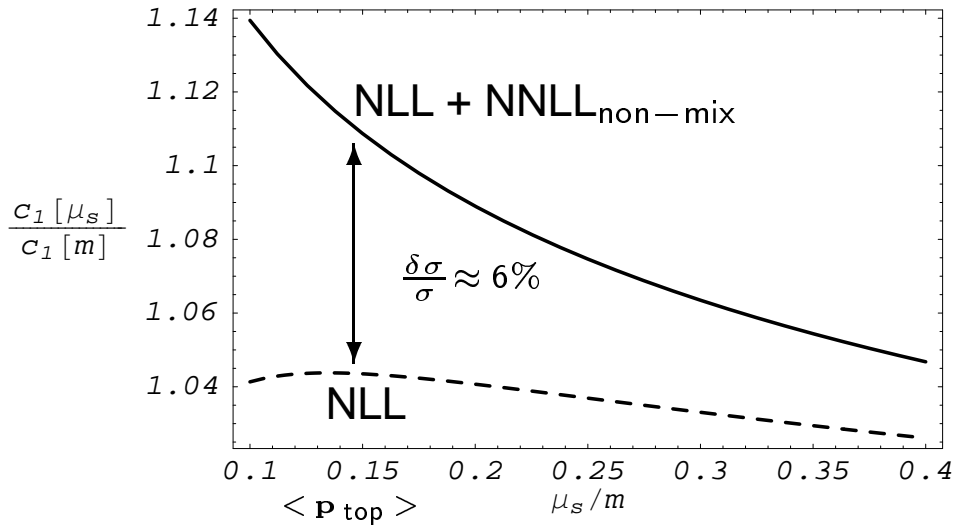
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missing \Rightarrow $V^{\text{NLL}}(\mu)$ needed!

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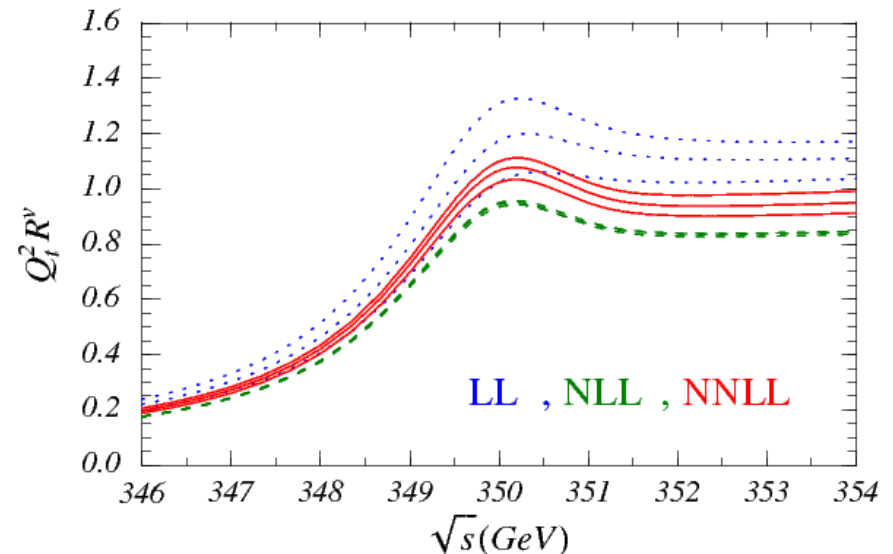
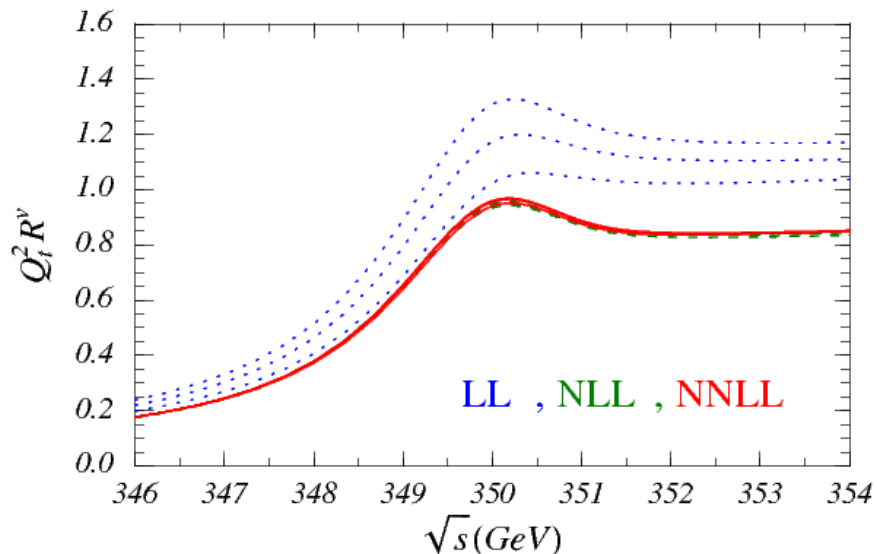
missing NNLL_{mix} contribution
may reduce theoretical error of σ_{tot}

\Rightarrow $V^{\text{NLL}}(\mu)$ needed!

Renormalization of the Potentials

NNLL_{nonmix} contribution to c_1 has large effect on σ_{tot} :

[Hoang]



NLL running of c_1 is used for **NNLL**

\Rightarrow small scale (ν) dependence

NNLL_{nonmix} of c_1 is included in **NNLL**

\Rightarrow large scale (ν) dependence

\Rightarrow large theoretical uncertainty ($\sim 6\%$) of σ_{tot}

Renormalization of the Potentials

NLL anomalous dimension of $c_1(\nu)$:

$$\mu \frac{\partial}{\partial \nu} \ln[c_1(\nu)] = -\frac{\mathcal{V}_c(\nu)}{16\pi^2} \left[\frac{\mathcal{V}_c(\nu)}{4} + \mathcal{V}_2(\nu) + \mathcal{V}_r(\nu) + \mathbf{S}^2 \mathcal{V}_s(\nu) \right] + \frac{1}{2} \mathcal{V}_k(\nu)$$

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NLL anomalous dimension of the potentials:

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- Relevant potentials for $c_1(\nu)$ at NNLL_{mix} , which are affected by NLL ultrasoft

renormalization: $\frac{\mathcal{V}_k \pi^2}{m\mathbf{k}}$, $\frac{\mathcal{V}_r(\mathbf{p}^2 + \mathbf{p}'^2)}{2m^2 \mathbf{k}^2}$, $\frac{\mathcal{V}_2}{m^2}$

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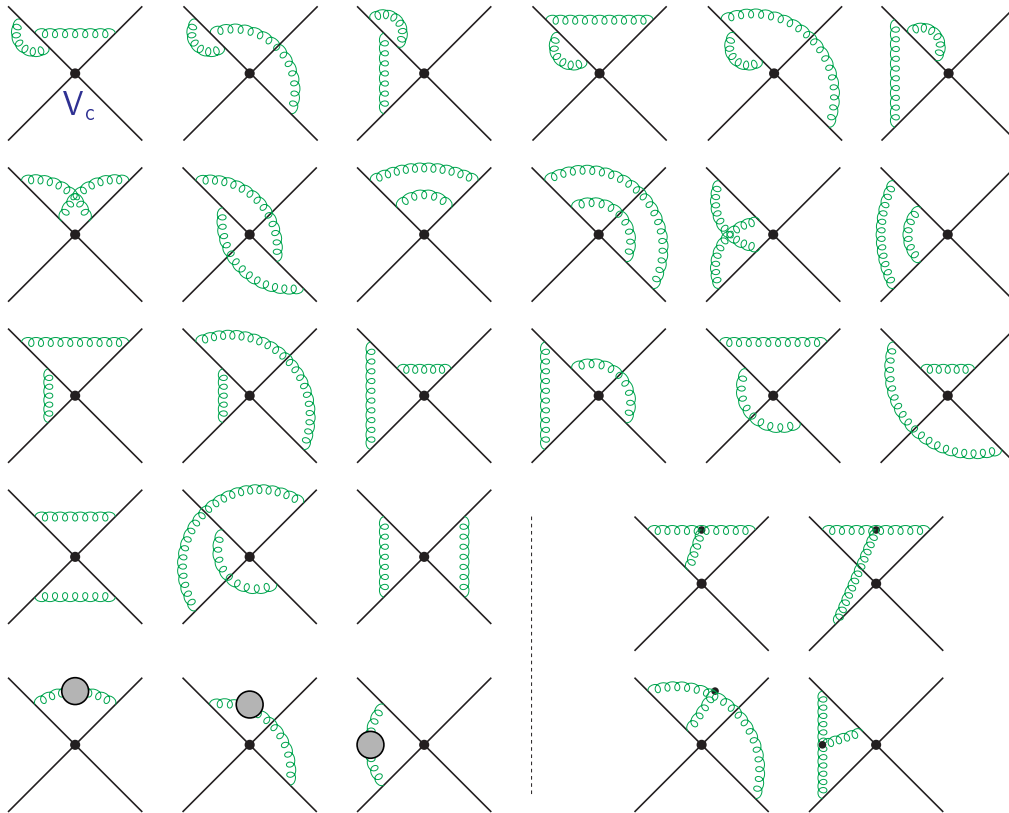
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- Cross check: ultrasoft $\mathcal{O}(v^0)$ -corrections to $V = 0$ \checkmark
follows from the form of the non. rel. Lagragian

(field redefinition: $\psi_{\mathbf{p}}^\dagger iD^0 \psi_{\mathbf{p}} \rightarrow \Phi_{\mathbf{p}}^\dagger i\partial^0 \Phi_{\mathbf{p}}$, $\psi_{\mathbf{p}}(\mathbf{x}) = \mathbf{W}(\mathbf{x}) \Phi_{\mathbf{p}}(\mathbf{x})$)

Renormalization of the Potentials

- Some 2-loop diagrams contributing to the NLL running of V_2 and V_r :



e.g.:

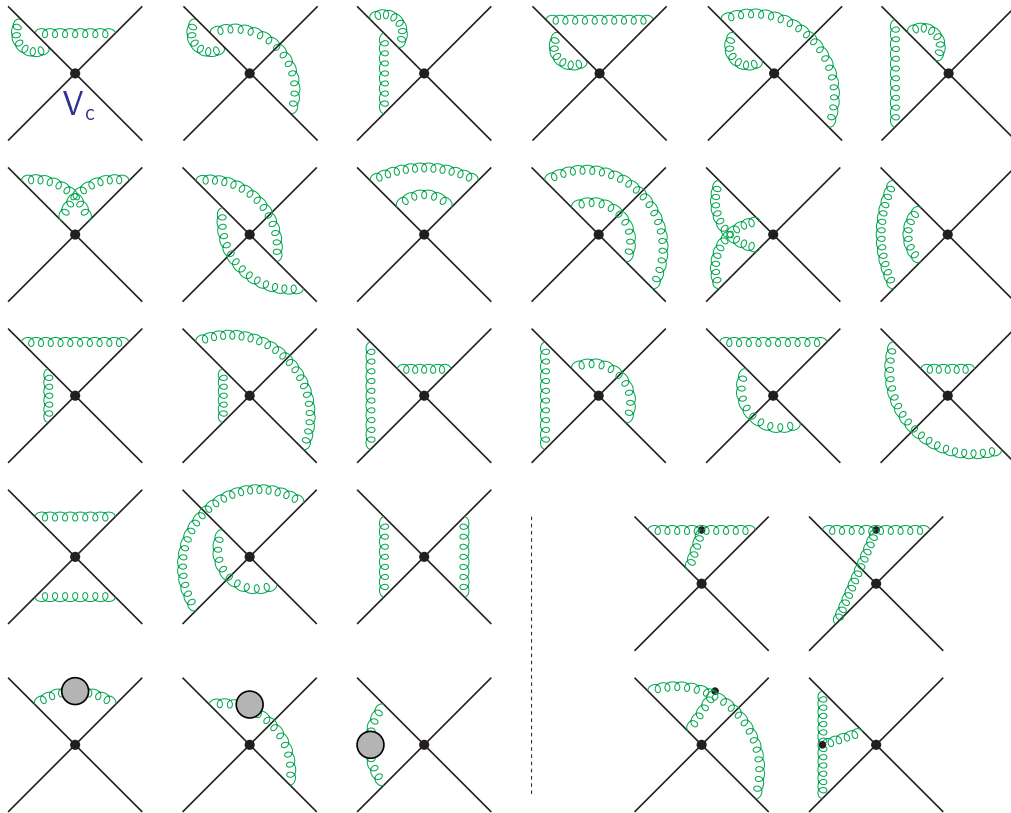
$$\Rightarrow \delta V_2^{2\text{loop}} \xrightarrow{\text{RGE}} V_2^{\text{NLL}}(\mu)$$

V_2^{NLL} and V_r^{NLL} complete ✓

[Stahlhofen, Hoang] to appear

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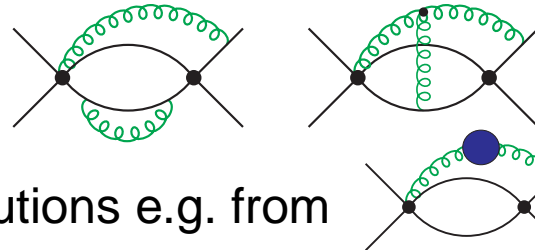
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- To V_k diagrams like



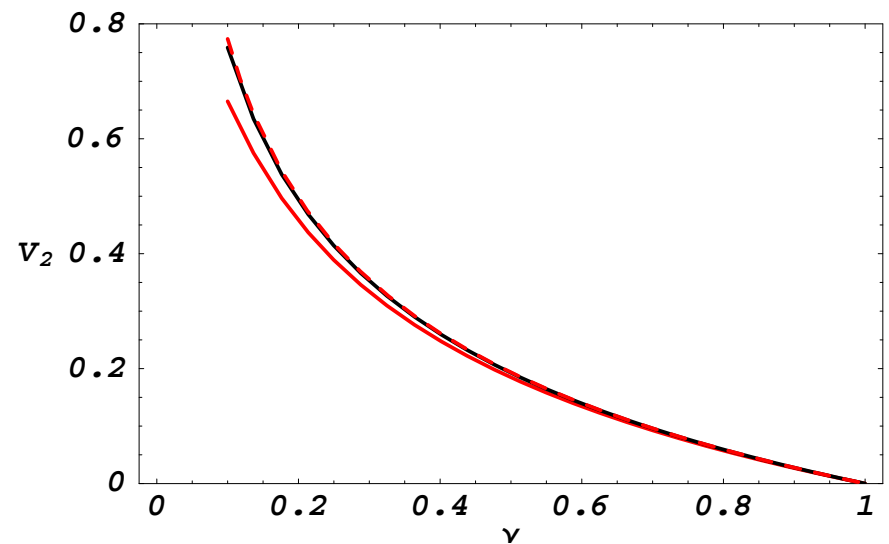
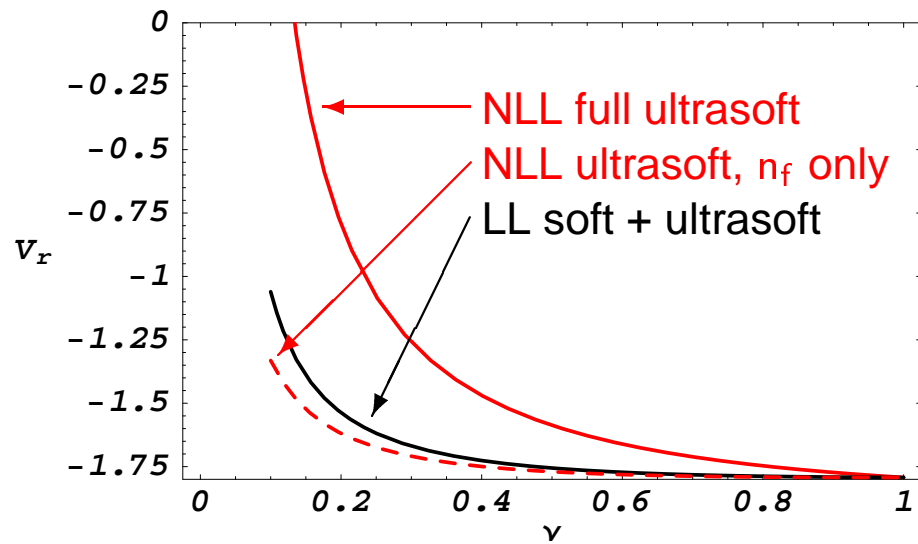
contribute: **work in progress!**

(fermionic (n_f) contributions e.g. from

complete ✓)

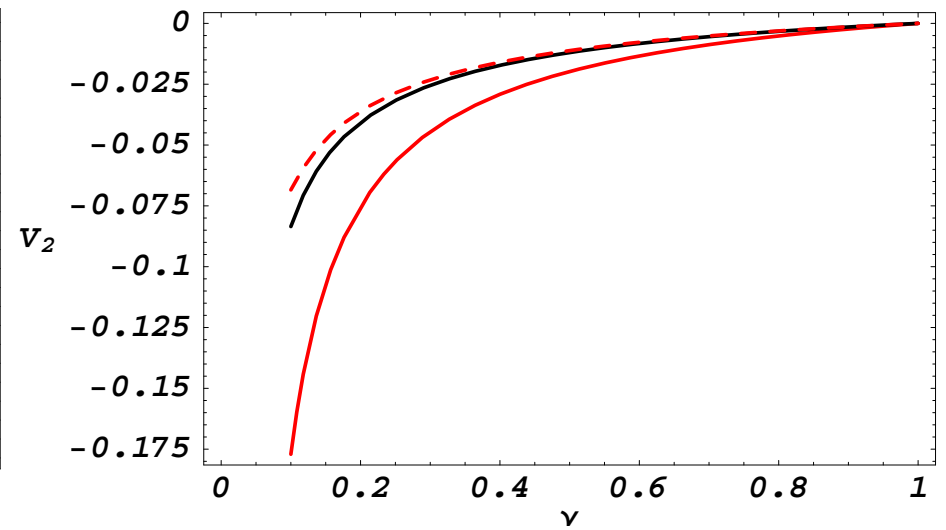
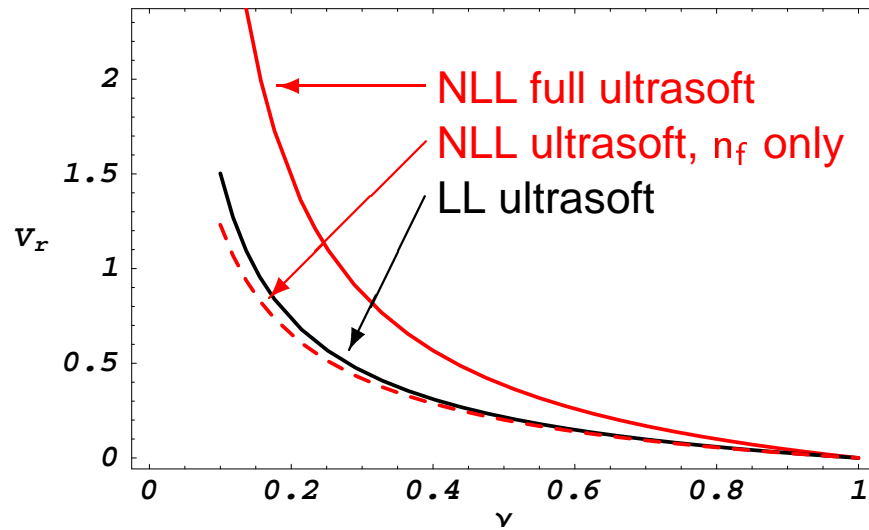
Status of Calculations

Preliminary results for $\frac{1}{m^2}$ potentials:



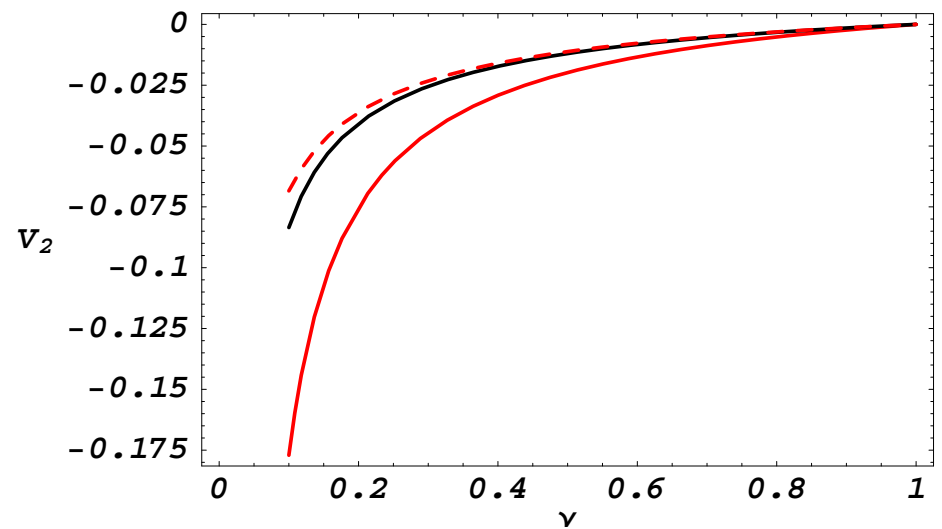
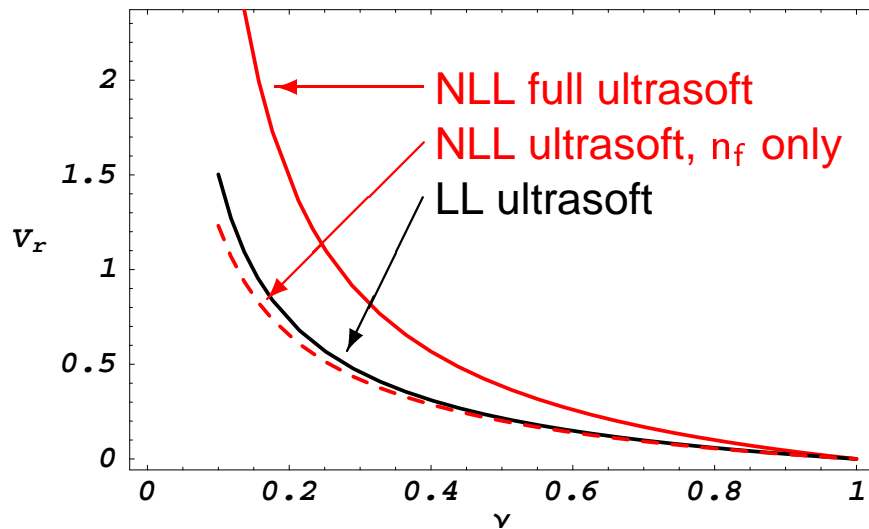
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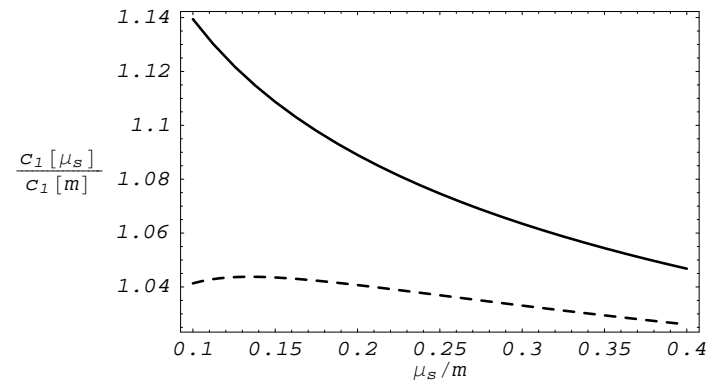
Preliminary results for $\frac{1}{m^2}$ potentials:



- Analysis shows that usoft LL \sim usoft NLL

\Rightarrow Big NNLL_{mix} contributions to c_1 expected

\Rightarrow may compensate $\text{NNLL}_{\text{nonmix}}$ and reduce ν dependence of c_1 !



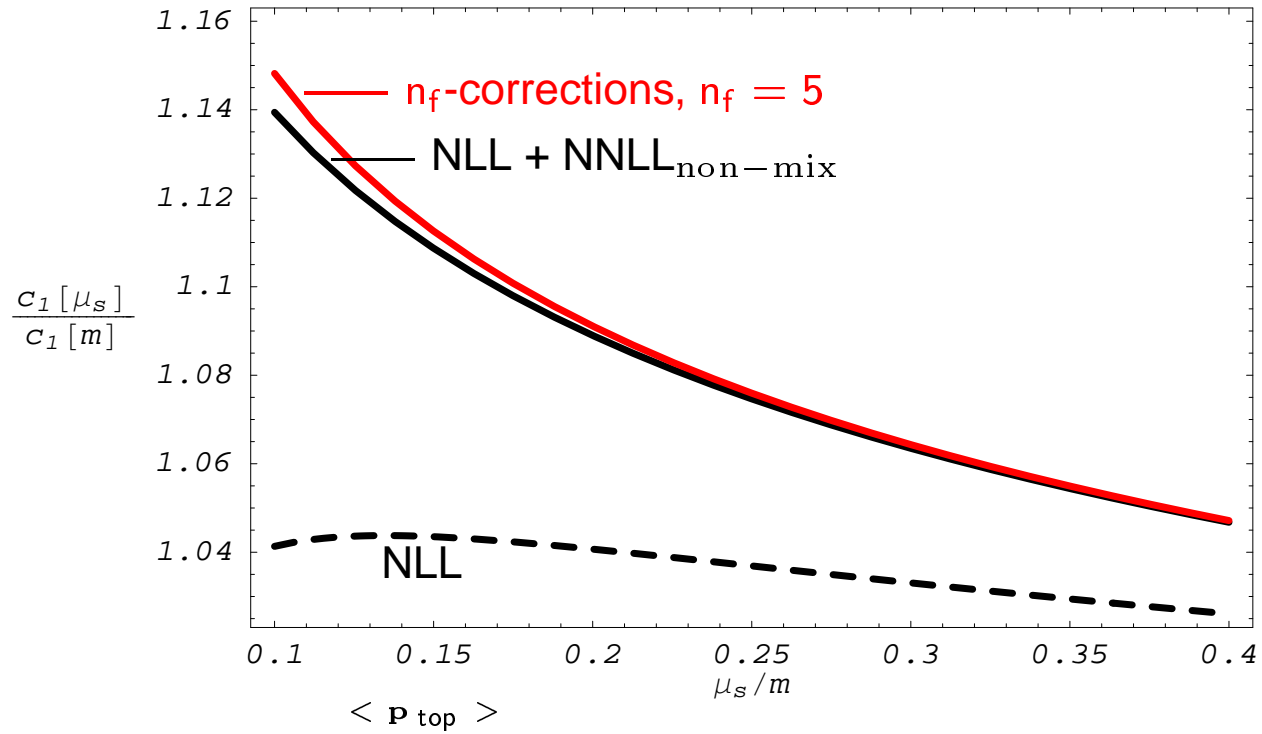
Summary

Ultrasoft NLL running of the potentials V_k , V_r , V_2 is essential for a precise prediction of $\sigma_{\text{tot}}(e^+ e^- \rightarrow t \bar{t})$ at threshold.

Current status of the calculation:

Contribution	V_k	V_r	V_2	V_s
soft + usoft LL	✓	✓	✓	✓
usoft NLL n_f	✓	✓	✓	0
full usoft NLL	—	✓	✓	0
soft NLL	—	—	—	✓

Old Result



Extra Formulae

$$\nu \frac{\partial}{\partial \nu} \ln[c_1(\nu)] = -\frac{\mathcal{V}_c^{(0)}(\nu)}{16\pi^2} \left[\frac{\mathcal{V}_c^{(0)}(\nu)}{4} + \mathcal{V}_2^{(2)}(\nu) + \mathcal{V}_r^{(2)}(\nu) + \mathbf{S}^2 \mathcal{V}_s^{(2)}(\nu) \right] + \frac{1}{2} \mathcal{V}_k^{(1)}(\nu) + \alpha_s^2(m\nu) [3\mathcal{V}_{k1}^{(1)}(\nu) + 2\mathcal{V}_{k2}^{(1)}(\nu)]$$

$$v \cong \alpha_s(mv) = \frac{4\pi}{\beta_0 \ln(m^2 v^2 / \Lambda_{\text{QCD}}^2)} \Rightarrow v \cong \alpha_s \cong 0.14$$

$$v \equiv \sqrt{\frac{\sqrt{s} - 2m_t}{m_t}} \rightarrow \sqrt{\frac{\sqrt{s} - 2m_t + i\Gamma_t}{m_t}} \quad [\text{Fadin, Khoze}]$$