GraphShot: a code to compute Feynman amplitudes

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The GraphShot project

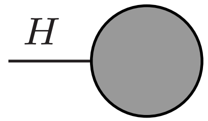
- A FORM code to generate and manipulate the amplitudes in the SM
- A link to FORTRAN libraries for numerical computation
- Authors: G.Passarino, S.Actis, C.Sturm, S.U.
- It is WORK IN PROGRESS (not yet available)

Let's discover the path to compute Feynman amplitudes ...

1. The Feynman rules

- The SM Lagrangian $\rightarrow$ normal rules for propagators and vertices
- Special rules:
  - Higgs vacuum expectation value

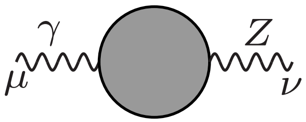
normal : 

$= \ 0$ special : 

$= \ 0$

  - Z-Photon exchange ($g \rightarrow g\,(1+\Gamma)$):

normal : $\Gamma = 0$

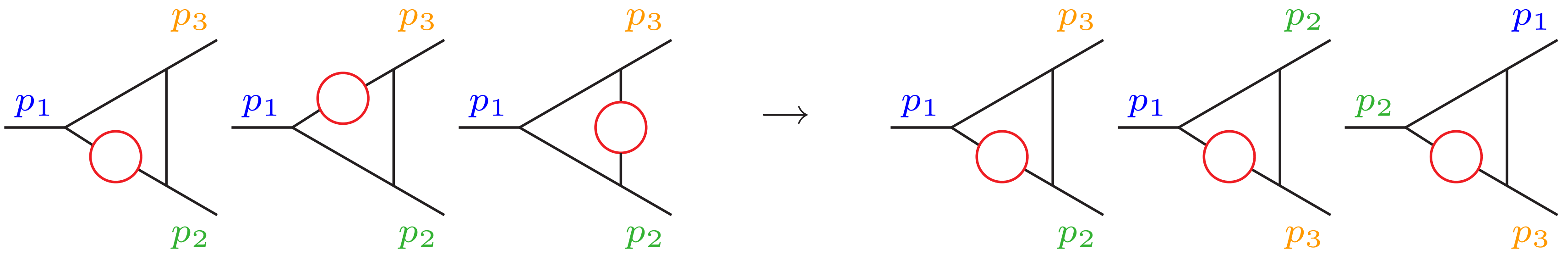
special : 

$= \ \mathcal{G}_d^{AZ}(p^2)\,\delta_{\mu\nu} + \mathcal{G}_{pp}^{AZ}(p^2)\,p_\mu p_\nu, \qquad \mathcal{G}_d^{AZ}(0) = 0$

- Renormalization $\rightarrow$ $\overline{MS}$ scheme
  - Counterterms for couplings, masses, fields, ...
  - Finite Feynman amplitudes

2. Generate the amplitude

- Group the diagrams into families, paying attention to:
  - Permutation of external legs



  - Combinatorial factors (Goldberg strategy)
- Combine the topologies and the Feynman rules
- Introduce projectors
- Compute the trace of Dirac matrices

$\Downarrow$

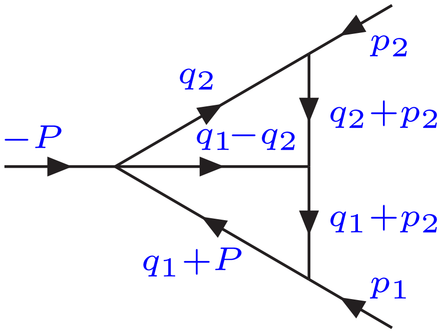
All loop momenta are contracted with other momenta

3. Reduction to Master Integrals

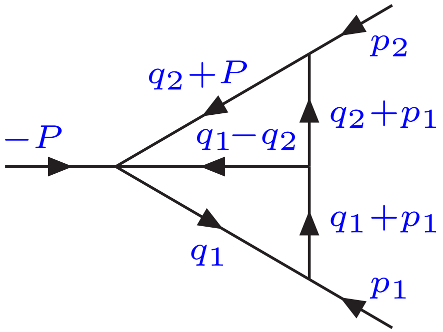
- Recursive application of:
  - Obvious reduction:

$$\frac{2\,q.p}{(q^2+m^2)\,[(q+p)^2+M^2]} = \frac{1}{q^2+m^2} - \frac{1}{(q+p)^2+M^2} - \frac{p^2-m^2+M^2}{(q^2+m^2)\,[(q+p)^2+M^2]}$$

  - Mapping on a fixed standard routing for loop momenta:

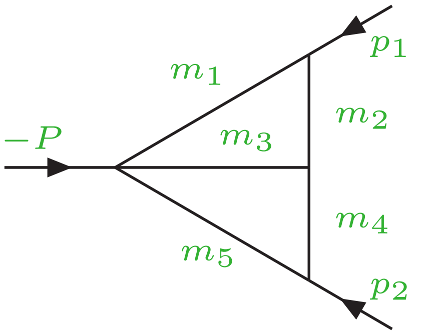


$\longrightarrow$

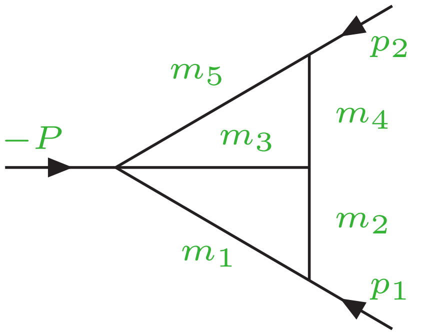


$$\begin{pmatrix} q_1 \to -q_1 - P \\ q_2 \to -q_2 - P \end{pmatrix}$$

  - Symmetrization:



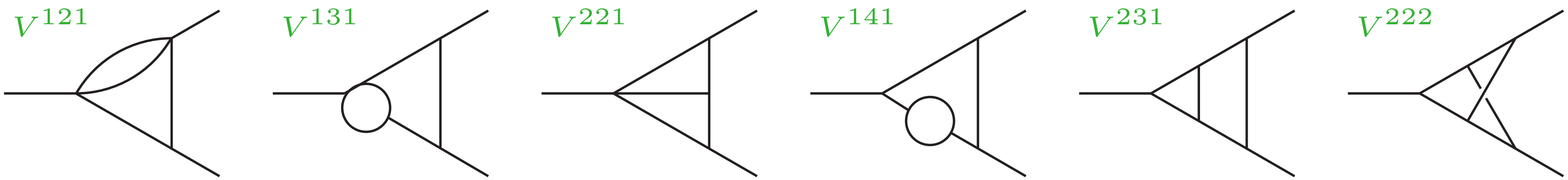
$\longrightarrow$



$$\begin{pmatrix} q_1 \to -q_2 - P \\ q_2 \to -q_1 - P \end{pmatrix}$$

- We end with integrals up to rank 3:
  - 1-loop functions
  - 2-loop self-energies (4 topologies)

  - 2-loop vertices (6 topologies)



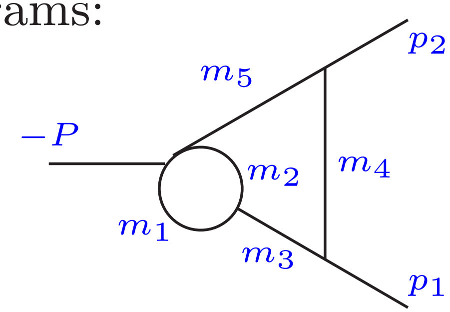
- Full scalarization $\longrightarrow$ possible for 1-loop diagrams and 2-loop self-energies
- For all, few scalar products are remaining

4. Analytical cancellations of divergences

Extraction of the UV poles

- 1-loop diagrams → trivial ($\Gamma(\epsilon/2)$)
- 2-loop diagrams:

$V^{131} =$



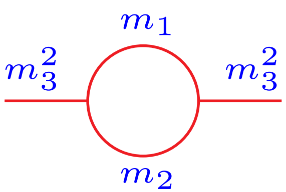
$$= \frac{1}{\pi^4} \int \frac{d^n q_1\, d^n q_2}{\underbrace{\underbrace{[1]\,[2]}_{x}\,[3]\,[4]\,[5]}_{y_1,y_2,y_3}},$$

$[1] = q_1^2 + m_1^2$
$[2] = (q_1 - q_2)^2 + m_2^2$
$[3] = q_2^2 + m_3^2$
$[4] = (q_2 + p_1)^2 + m_4^2$
$[5] = (q_2 + P)^2 + m_5^2$

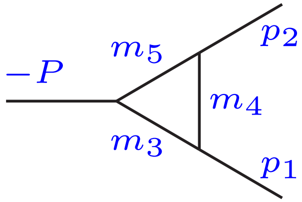
$$= C_\epsilon \int_0^1 dx \int dS_3(y_1, y_2, y_3)\, [x\,(1-x)]^{-\epsilon/2}\, (1-y_1)^{\epsilon/2-1}\, V^{-1-\epsilon}$$

- Overall divergency → trivial ($\Gamma(\epsilon)$)
- Singularities coming from sub-loops → hidden in the integrand
- The single pole can always be expressed in terms of 1-loop functions.

$V^{131} =$



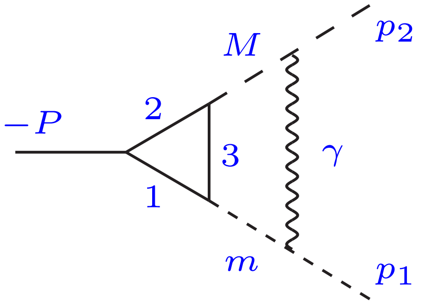
$\times$



$+$ finite part.

Infrared singularities

V_{IR}^{231} =

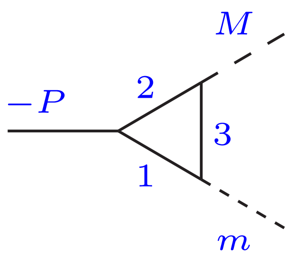


$$= \frac{1}{\pi^4}\int \frac{d^n q_1\, d^n q_2}{\underbrace{\underbrace{[1]\,[2]\,[3]}_{x_1,x_2}\,[4]\,[5]\,[6]}_{y_1,y_2,y_3}},$$

$$[1] = q_1^2 + m_1^2$$
$$[2] = (q_1 + p_1)^2 + m_2^2$$
$$[3] = (q_1 - q_2)^2 + m_3^2$$
$$[4] = (q_2 + p_1)\cdot(q_2 - p_1)$$
$$[5] = (q_2 + p_1)^2$$
$$[6] = (q_2 + p_1)\cdot(q_2 + p_1 + 2p_2)$$

- Small momenta singularity $(q_2 + p_1 = 0)$
- Endpoint singularites in parametric space $(y_1 = y_2 = y_3 = 0)$:
  - Keep $\epsilon \neq 0$
  - Use hypergeometric functions

V_{IR}^{231} =



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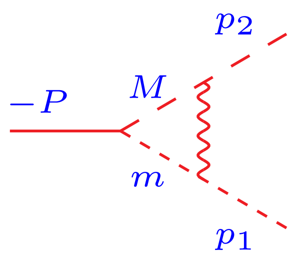
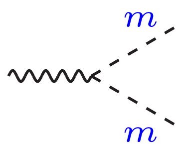


+ finite part.

Collinear divergencies

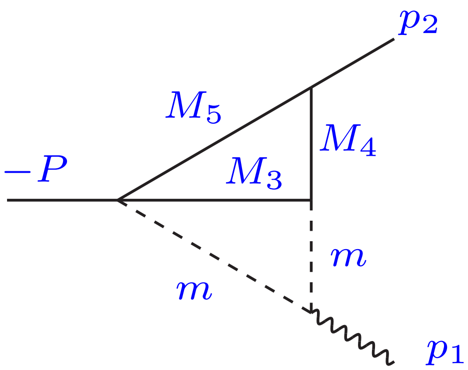
- They come from the coupling of light particles (m) with massless particles



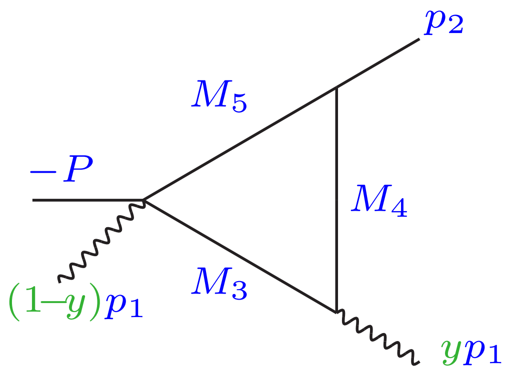
- Single divergency: Subtraction method

$$\int_0^1 dx \frac{1}{xa(x)+m^2b(x)} = \int_0^1 dx \left[\frac{1}{xa(x)+m^2b(x)} - \frac{1}{xa(0)+m^2b(0)} + \frac{1}{xa(0)+m^2b(0)}\right]$$

$$\sim \int_0^1 dx \frac{1}{x}\left[\frac{1}{a(x)} - \frac{1}{a(0)}\right] - \frac{1}{a(0)} \ln \frac{m^2 b(0)}{a(0)}$$



$$= \ln \frac{m^2}{s} \int_0^1 dy$$



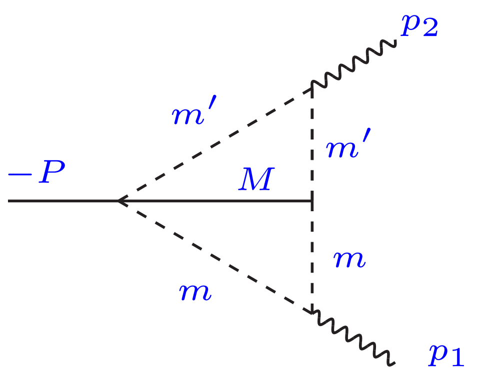
$+$ finite part

- Double divergency: Double subtraction

$$\int_0^1 dxdy \frac{1}{xya(x,y)+\lambda b(x,y)} = \int_0^1 dxdy \Bigg\{ \frac{1}{xya(x,y)+\lambda b(x,y)}\Bigg|_{x,y} + \frac{1}{xya(x,0)+\lambda b(x,0)}\Bigg|_{x} + \frac{1}{xya(0,y)+\lambda b(0,y)}\Bigg|_{y} + \frac{1}{xya(0,0)+\lambda b(0,0)} \Bigg\}, \qquad \lambda \to 0$$

$$f(z)|_z = f(z) - f(z)|_{z^2=\lambda z=0}$$

- First term $\longrightarrow$ set $\lambda = 0$
- Second (third) term $\longrightarrow$ integrate in y (x) $\longrightarrow$ $\ln(\lambda)$
- Last term $\longrightarrow$ integrate in x and y $\longrightarrow$ $\ln^2(\lambda)$



$$= \ln\frac{m^2}{s}\ln\frac{m'^2}{s}\mathrm{Li}_2\left(\frac{s}{M^2}\right) + \left(\ln\frac{m^2}{s} + \ln\frac{m'^2}{s}\right)\left[\mathrm{Li}_3\left(\frac{s}{M^2}\right) + 2\,S_{12}\left(\frac{s}{M^2}\right) - \ln\frac{M^2}{s}\mathrm{Li}_2\left(\frac{s}{M^2}\right)\right] + \text{finite part}$$

5. Numerical computation

Write the **finite part** in one of the following forms:

1) $\int dx \, \frac{Q(x)}{V(x)}$ $\qquad V(x)$ polinomial positive definite

2) $\frac{1}{B} \int dx \, Q(x) \, \ln^n V(x)$ $\qquad B$ constant $\neq 0$.

3) $\int dx \, \frac{Q(x)}{V(x)} \, f\left(\frac{V(x)}{P(x)}\right)$ $\qquad f(0) = 0,$ $\qquad f(x) = ln^n(1+x),\ Li_n(x),\ S_{n,p}(x)$

(Improved) Bernstein-Sato-Tkachov (BST) approach

$$V^{\mu}(z) = \frac{1}{B}\left(1 + \frac{\mathcal{P}^t \, \partial_z}{\mu + 1}\right) V^{\mu+1}(z), \qquad \mathcal{P} = -\frac{z - Z}{2}$$

$$V(z) = z^t H z + 2K^t z + L, \qquad Z = -K^t H^{-1}, \qquad B = L - K^t H^{-1} K$$

- A new very usefull relation is:

$$\frac{1}{V} \ln\left(1 - \frac{A}{V}\right) = \frac{1}{B}\left[\frac{1}{A} \ln\left(1 - \frac{A}{V}\right)(1 + \mathcal{P}_1^t \, \partial_z) \, A + \mathcal{P}_1^t \, \partial_z \, \mathrm{Li}_2\left(\frac{A}{V}\right)\right]$$

A new BST relation

$$V(z) = z^t\, H\, z + 2\, K^t\, z + L = (z^t - Z^t)\, H\, (z - Z) + B = Q(z) + B$$

It can be easily proven that:

$$\mathcal{P}^t \partial_z\, Q(z) = -Q(z), \quad \mathcal{P} = -\frac{z-Z}{2}, \qquad V^\mu(z) = \left(\beta - \mathcal{P}^t\, \partial_z\right) \int_0^1 dy\, y^{\beta-1} \left[Q(z)\, y + B\right]^\mu$$

If $\mu = -1$ and $\beta = 1$

$$V^{-1} = \left(1 - \mathcal{P}^t\, \partial_z\right) \frac{1}{Q}\, \ln\left(1 + \frac{Q}{B}\right)$$

Example: scalar 3-point function

$$C_0 = \sum_{i=1}^{3} \frac{a_i}{2} \int_0^1 dx_1\, \frac{1}{V[i](x_1) - B}\, \ln \frac{V[i](x_1)}{B}, \qquad a_1 = 1 - Z_1, \quad a_2 = Z_1 - Z_2, \quad a_3 = Z_2.$$

$V[i]$ is the polynomial of the two-point function obtained by shrinking to a point the i^{th} propagator.

State of the art

- Algebraic manipulation → implemented for 1-,2-,3-point 1-,2-loop functions
- Fortran codes → available for 1-,2-,3-point 1-,2-loop functions, massive and IR (link to Graphshot to be done)
- Extraction of collinear logs → partially done
- Future extension → 1-loop multi-leg processes

Applications

- First (partial) application: recent computation of $\sin^2\theta_{\rm eff}$
- Process under examination: $H \to \gamma\,\gamma$