DRED applied to QCD at 3 and 4 loops

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Outline

- Motivation
- DRED Framework
- QCD: 3- and 4-loop results for $\beta^{\overline{\mathrm{DR}}}$ and $\gamma_m^{\overline{\mathrm{DR}}}$
- Conclusions

Motivation

- Precision calculations of the LHC- and ILC-observables

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  - DRED applied to non supersymmetric theories
  - QCD as the low energy effective theory of SUSY-QCD

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- Precision calculations of the LHC- and ILC-observables
- QCD & DRED:
  - Use of Ward identities for SUSY Yang-Mills theories
  - DRED applied to non supersymmetric theories
  - QCD as the low energy effective theory of SUSY-QCD
- Removal of DRED inconsistencies [W. Siegel '80], [D. Stöckinger '05]
  - possible SUSY violation at HO

    [L. Avdeev, G. Chochia, A. Vladimirov '81], [I. Jack and D. R. T. Jones '97]
  - SUSY preserved in all present 1- and 2-loop checks

    [W. Hollik and D. Stöckinger '05]

Framework

Quasi-4-dim. space (Q4S): $4 = d \oplus 4 - d$

- *Quasi*-4-dim metric tensor: $G_{\mu\nu} = g_{\mu\nu} + \tilde{g}_{\mu\nu}$

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$$G_{\mu\alpha}\, G_{\mu\alpha} = 4\,, \quad G_{\mu\alpha}\, g_{\alpha\nu} = g_{\mu\nu}\,, \quad G_{\mu\alpha}\, \tilde{g}_{\alpha\nu} = \tilde{g}_{\mu\nu}$$

$$g_{\mu\mu} = d\,, \quad g_{\mu\alpha}\, g_{\alpha\nu} = g_{\mu\nu}\,, \quad g_{\mu\alpha}\, \tilde{g}_{\alpha\nu} = 0$$

$$\tilde{g}_{\mu\mu} = 4 - d\,, \quad \tilde{g}_{\mu\alpha}\, \tilde{g}_{\alpha\nu} = \tilde{g}_{\mu\nu}$$

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Γ_μ do not satisfy 4-dim. Fierz relations

$$\gamma_\mu = g_{\mu\nu}\,\Gamma_\nu\,, \qquad \tilde{\gamma}_\mu = \tilde{g}_{\mu\nu}\,\Gamma_\nu$$

$$\{\Gamma_\mu\,,\Gamma_\nu\} = 2\,G_{\mu\nu}\,\mathbb{I}\,, \quad \mathrm{Tr}\,\mathbb{I} = 2^2 = 4\,,$$

$$\{\gamma_\mu\,,\gamma_\nu\} = 2\,g_{\mu\nu}\,\mathbb{I}\,, \quad \{\tilde{\gamma}_\mu\,,\tilde{\gamma}_\nu\} = 2\,\tilde{g}_{\mu\nu}\,\mathbb{I}\,, \quad \{\gamma_\mu\,,\tilde{\gamma}_\nu\} = 0\,.$$

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$$\begin{aligned} \tilde{\partial}_\mu &\equiv \tilde{g}_{\mu\nu}\partial_\nu = 0 \\ \tilde{p}_\mu &= 0 \end{aligned}$$

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- the number of field components unchanged
  - 4-dim gluon field: $A^a_\mu = V^a_\mu + S^a_\mu\,,$

$$\begin{aligned} V^a_\mu &= g_{\mu\nu}\,A^a_\nu = \quad d\text{- dim. vector} \\ S^a_\mu &= \tilde{g}_{\mu\nu}\,A^a_\nu = \quad \varepsilon \text{ scalar} \end{aligned}$$

under gauge transformations

Framework(2)

$$\mathcal{L} = \mathcal{L}^n + \mathcal{L}^\varepsilon$$

- $\mathcal{L}^n$ same as in DREG
- $\mathcal{L}^\varepsilon$ new contribution due to ε-scalars
- distinguish between real and evanescent couplings

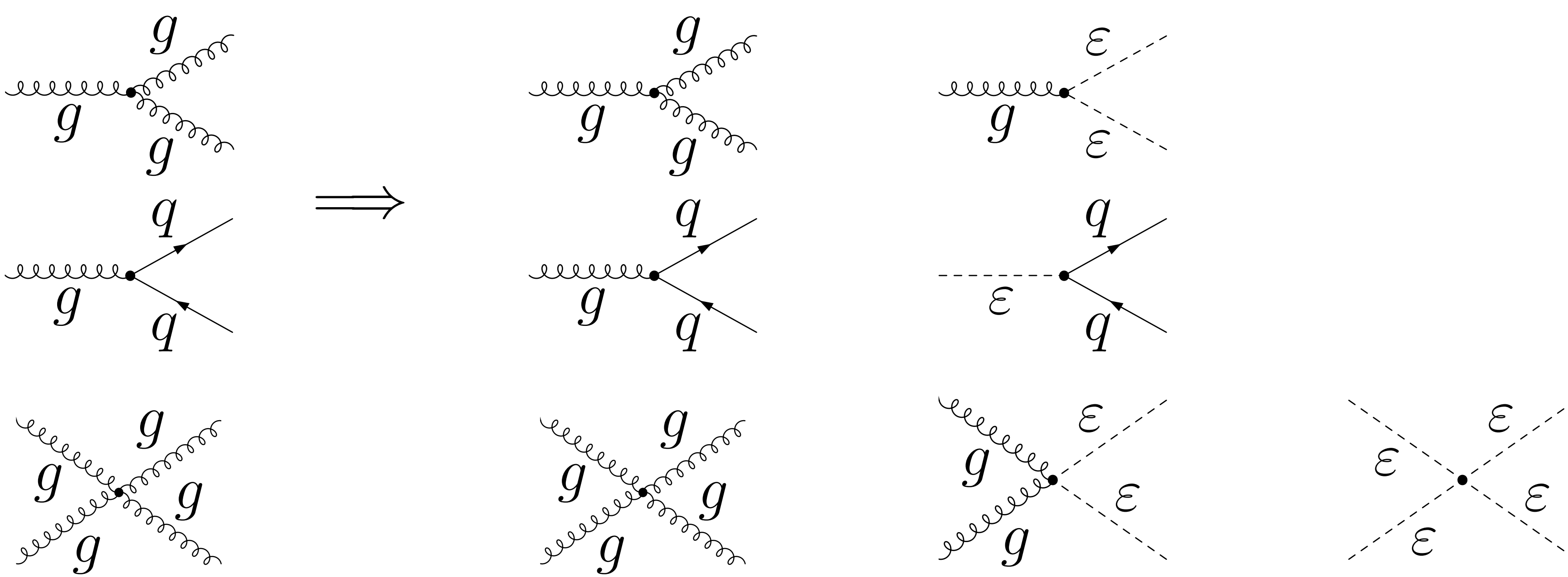
$$\mathcal{L}^n = -\frac{1}{4}V^{a,\mu\nu}V^a_{\mu\nu} - \frac{(\partial^\mu V^a_\mu)^2}{2(1-\xi)} + \bar{c}^a\partial^\mu\nabla^{ab}_\mu c^b + i\bar{\psi}^\alpha\gamma^\mu\nabla^{\alpha\beta}_\mu\psi^\beta$$

$$\mathcal{L}^\varepsilon = \frac{1}{2}(\nabla^{ab}_\mu S^b_\nu)^2 - g\bar{\psi}\gamma_\nu T^a\psi S^a_\nu - \frac{1}{4}g^2 f^{abc}f^{ade}S^b_\nu S^c_\rho S^d_\nu S^e_\rho$$

  - S_ν: Yukawa-type (α_e) and quartic self-couplings (η_r)
  - $f-f$ structure not preserved under renormalization
  - 3 new couplings η_r allowed

Feynman Rules

- ε-scalars treated as real particles
- new Feynman diagrams/rules for the ε-scalars



- real and evanescent couplings renormalize differently

Charge Renormalization Constants

- gauge coupling α_s: $$Z_s = \frac{\tilde{Z}_1}{\tilde{Z}_3\sqrt{Z_3}} = \frac{Z_1}{Z_2\sqrt{Z_3}}$$

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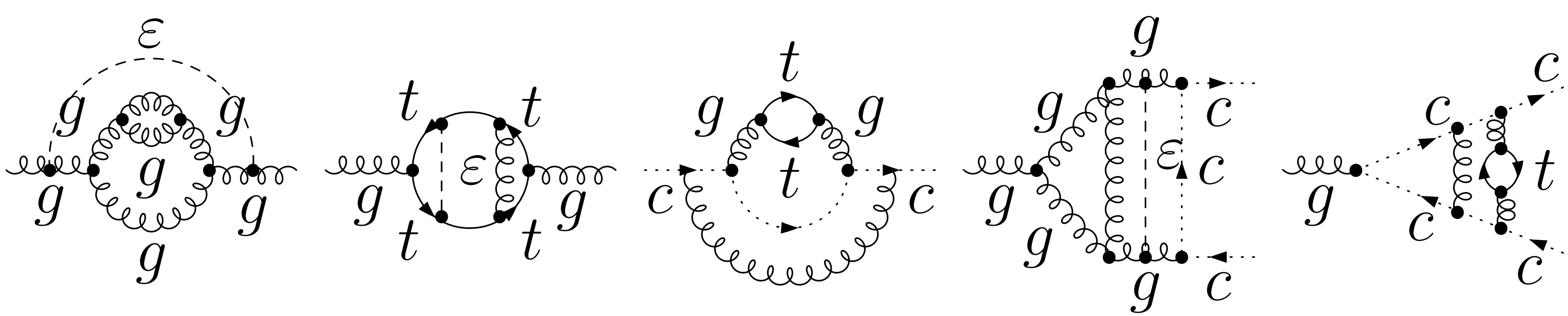

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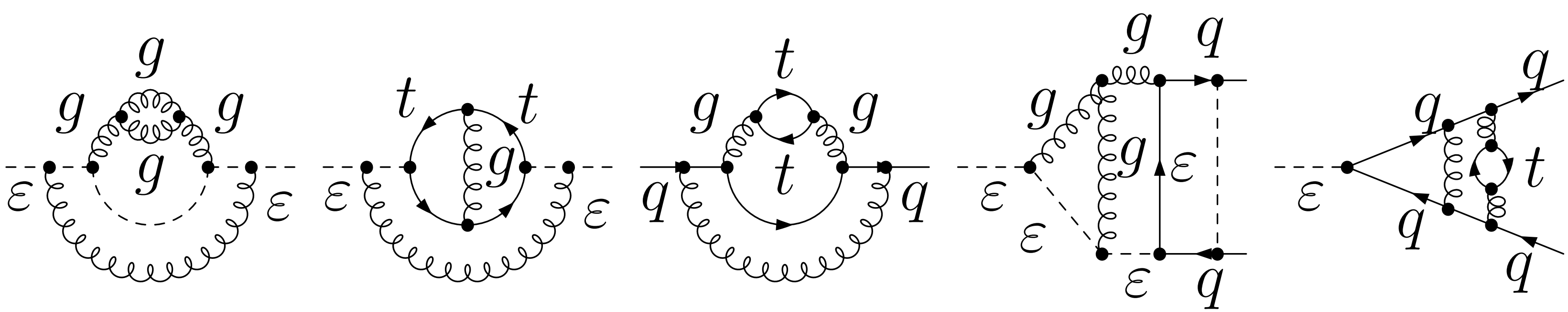

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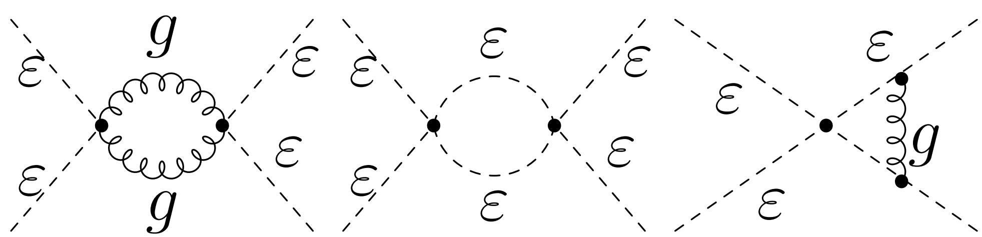

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- quartic ε-scalar coupling η_r: $Z_{\lambda_r} = \frac{\sqrt{Z_1^r}}{Z_3^\varepsilon}$
- Renormalization conditions: all Green's functions finite $\Rightarrow$ Unitarity is maintained.

β-functions within DRED

- Dimensional Reduction $\oplus$ Minimal Subtraction $\overline{\mathrm{DR}}$

$$\beta_s^{\overline{\mathrm{DR}}}(\alpha_s^{\overline{\mathrm{DR}}}, \alpha_e, \{\eta_r\}) = \mu^2 \frac{\mathrm{d}}{\mathrm{d}\mu^2} \frac{\alpha_s^{\overline{\mathrm{DR}}}}{\pi}$$

$$= -\left[\epsilon \frac{\alpha_s^{\overline{\mathrm{DR}}}}{\pi} + 2 \frac{\alpha_s^{\overline{\mathrm{DR}}}}{Z_s^{\overline{\mathrm{DR}}}} \left(\frac{\partial Z_s^{\overline{\mathrm{DR}}}}{\partial \alpha_e} \beta_e + \sum_r \frac{\partial Z_s^{\overline{\mathrm{DR}}}}{\partial \eta_r} \beta_{\eta_r} \right)\right] \left(1 + 2 \frac{\alpha_s^{\overline{\mathrm{DR}}}}{Z_s^{\overline{\mathrm{DR}}}} \frac{\partial Z_s^{\overline{\mathrm{DR}}}}{\partial \alpha_s^{\overline{\mathrm{DR}}}}\right)^{-1}$$

$$\beta_e(\alpha_s^{\overline{\mathrm{DR}}}, \alpha_e, \{\eta_r\}) = \mu^2 \frac{\mathrm{d}}{\mathrm{d}\mu^2} \frac{\alpha_e}{\pi}$$

$$= -\left[\epsilon \frac{\alpha_e}{\pi} + 2 \frac{\alpha_e}{Z_e} \left(\frac{\partial Z_e}{\partial \alpha_s^{\overline{\mathrm{DR}}}} \beta_s^{\overline{\mathrm{DR}}} + \sum_r \frac{\partial Z_e}{\partial \eta_r} \beta_{\eta_r} \right)\right] \left(1 + 2 \frac{\alpha_e}{Z_e} \frac{\partial Z_e}{\partial \alpha_e}\right)^{-1}$$

$$\beta_{\eta_r}(\alpha_s^{\overline{\mathrm{DR}}}, \alpha_e, \{\eta_r\}) = \mu^2 \frac{\mathrm{d}}{\mathrm{d}\mu^2} \frac{\eta_r}{\pi}$$

$$= -\left[\epsilon \frac{\eta_r}{\pi} + 2 \frac{\eta_r}{Z_{\lambda_r}} \left(\frac{\partial Z_{\lambda_r}}{\partial \alpha_s^{\overline{\mathrm{DR}}}} \beta_s^{\overline{\mathrm{DR}}} + \frac{\partial Z_{\lambda_r}}{\partial \alpha_e} \beta_e + \sum_{r' \neq r} \frac{\partial Z_{\lambda_r}}{\partial \eta_{r'}} \beta_{\eta_{r'}} \right)\right] \left(1 + 2 \frac{\eta_r}{Z_{\lambda_r}} \frac{\partial Z_{\lambda_r}}{\partial \eta_r}\right)^{-1}$$

3-loop $\beta^{\overline{\rm DR}}$-function

- Explicit computation
  - Z_s to 3-loops
    - disagreement with the existing result [Z. Bern et al. ’02]
  - Z_e to 1-loop [I. Jack et al. ’94], [L.Avdeev & M.Kalmykov’97]
  - $\beta_s^{\overline{\rm DR}}$ independent of η_r up to 3-loops

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$$\beta_{\mathbf{g}}^{\mathrm{dred},3l}(\alpha_{\mathbf{s}}^{\overline{\mathrm{DR}}},\alpha_{\mathbf{e}}) = \left[\frac{\alpha_s^{\overline{\mathrm{DR}}}}{\pi}\right]^3 \frac{\alpha_e}{\pi}\frac{3}{16}C_F^2 T n_f + \left[\frac{\alpha_s^{\overline{\mathrm{DR}}}}{\pi}\frac{\alpha_e}{\pi}\right]^2 C_F T n_f \left[\frac{C_A}{16} - \frac{C_F}{8} - \frac{T n_f}{16}\right]$$

$$- \left[\frac{\alpha_s^{\overline{\mathrm{DR}}}}{\pi}\right]^4 \left[\frac{3115}{3456}C_A^3 - \frac{1439}{1728}C_A^2 T n_f + \frac{1}{32}C_F^2 T n_f - \frac{193}{576}C_A C_F T n_f + \frac{79}{864}C_A T^2 n_f^2 + \frac{11}{144}C_F T^2 n_f^2\right]$$

3-loop $\beta_s^{\overline{\rm DR}}$-function (2)

- Indirect computation

$$\beta_s^{\overline{\rm DR}} = \beta_s^{\overline{\rm MS}} \frac{\partial \alpha_s^{\overline{\rm DR}}}{\partial \alpha_s^{\overline{\rm MS}}} + \beta_e \frac{\partial \alpha_s^{\overline{\rm DR}}}{\partial \alpha_e} + \ldots$$

  - 2-loop conversion relation $\alpha_s^{\overline{\rm MS}} \leftrightarrows \alpha_s^{\overline{\rm DR}}$

$$\frac{\alpha_s^{\overline{\rm DR}}}{\alpha_s^{\overline{\rm MS}}} = 1 + \frac{\alpha_s^{\overline{\rm MS}}}{\pi}\frac{C_A}{12} + \left[\frac{\alpha_s^{\overline{\rm MS}}}{\pi}\right]^2 \frac{11}{72}C_A^2 - \frac{\alpha_s^{\overline{\rm MS}}}{\pi}\frac{\alpha_e}{\pi}\frac{1}{8}C_F T n_f$$

  - $\alpha_e = \alpha_s^{\overline{\rm DR}}$ QCD: [Z.Bern et al.'02]
    SUSY-QCD: [R. Harlander, L.M., M. Steinhauser'05]

- $\alpha_e \neq \alpha_s^{\overline{\rm DR}}$ proves equivalence of **DRED** and **DREG** at 3-loops

4-loop $\beta^{\overline{\rm DR}}$ - function

- Indirect computation

$$\beta_s^{\overline{\rm DR}} = \beta_s^{\overline{\rm MS}} \frac{\partial \alpha_s^{\overline{\rm DR}}}{\partial \alpha_s^{\overline{\rm MS}}} + \beta_e \frac{\partial \alpha_s^{\overline{\rm DR}}}{\partial \alpha_e} + \sum_r \beta_{\eta_r} \frac{\partial \alpha_s^{\overline{\rm DR}}}{\partial \eta_r}$$

  - 4-loop $\beta_s^{\overline{\rm MS}}$ known [T. van Ritbergen et al'97, M. Czakon'04]
  - 2-loop β_e computed
  - only $\mathcal{O}(\alpha_s)$ of β_{η_r} needed
  - 3-loop conversion relation $\alpha_s^{\overline{\rm MS}} \leftrightarrows \alpha_s^{\overline{\rm DR}}$ computed

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  - only $\mathcal{O}(\alpha_s)$ of β_{η_r} needed
  - 3-loop conversion relation $\alpha_s^{\overline{\rm MS}} \leftrightarrows \alpha_s^{\overline{\rm DR}}$ computed

$$\left(\frac{\alpha_s^{\overline{\rm DR}}}{\alpha_s^{\overline{\rm MS}}}\right)_{3l} = \left(\frac{\alpha_s^{\overline{\rm MS}}}{\pi}\right)^3 \left(\frac{3049}{384} - \frac{179}{864} n_f\right) + \frac{\left(\alpha_s^{\overline{\rm MS}}\right)^2}{\pi^3} \left(-\eta_1 \frac{9}{256} + \eta_2 \frac{15}{32} + \eta_3 \frac{3}{128} - \alpha_e \frac{887}{1152} n_f\right)$$
$$+ \frac{\alpha_s^{\overline{\rm MS}}}{\pi^3} \left[\eta_1^2 \frac{27}{256} - \eta_2^2 \frac{15}{16} - \eta_1 \eta_3 \frac{9}{64} + \eta_3^2 \frac{21}{128} + \alpha_e^2 \left(\frac{43}{864} n_f + \frac{19}{1152} n_f^2\right)\right]$$

4-loop $\beta^{\overline{\mathrm{DR}}}$ - function

- Indirect computation

$$\beta_s^{\overline{\mathrm{DR}}} = \beta_s^{\overline{\mathrm{MS}}} \frac{\partial \alpha_s^{\overline{\mathrm{DR}}}}{\partial \alpha_s^{\overline{\mathrm{MS}}}} + \beta_e \frac{\partial \alpha_s^{\overline{\mathrm{DR}}}}{\partial \alpha_e} + \sum_r \beta_{\eta_r} \frac{\partial \alpha_s^{\overline{\mathrm{DR}}}}{\partial \eta_r}$$

  - 4-loop $\beta_s^{\overline{\mathrm{MS}}}$ known [T. van Ritbergen et al'97, M. Czakon'04]
  - 2-loop β_e computed
  - only $\mathcal{O}(\alpha_s)$ of β_{η_r} needed
  - 3-loop conversion relation $\alpha_s^{\overline{\mathrm{MS}}} \leftrightarrows \alpha_s^{\overline{\mathrm{DR}}}$ computed

- 4-loop order result:

$$\beta_s^{\overline{\mathrm{DR}}} = -\epsilon \frac{\alpha_s^{\overline{\mathrm{DR}}}}{\pi} - \sum_{i,j,k,l,m} \beta_{ijklm}^{\overline{\mathrm{DR}}} \left(\frac{\alpha_s^{\overline{\mathrm{DR}}}}{\pi}\right)^i \left(\frac{\alpha_e}{\pi}\right)^j \left(\frac{\eta_1}{\pi}\right)^k \left(\frac{\eta_2}{\pi}\right)^l \left(\frac{\eta_3}{\pi}\right)^m$$

$$\beta^{\overline{\rm DR}}_{50000} = \frac{\beta_3}{256} + \frac{166861}{6144} - \frac{9109}{6912} n_f + \frac{457}{20736} n_f^2 \,, \qquad \beta^{\overline{\rm DR}}_{41000} = -\frac{1667}{512} n_f + \frac{145}{2304} n_f^2 \,,$$

$$\beta^{\overline{\rm DR}}_{32000} = -\frac{409}{6912} n_f + \frac{1303}{4608} n_f^2 \,, \qquad \beta^{\overline{\rm DR}}_{23000} = \frac{5}{1296} n_f - \frac{49}{3456} n_f^2 - \frac{19}{2304} n_f^3 \,,$$

$$\beta^{\overline{\rm DR}}_{40100} = -\frac{171}{512} + \frac{3}{512} n_f \,, \qquad \beta^{\overline{\rm DR}}_{40010} = \frac{285}{64} - \frac{5}{64} n_f \,, \qquad \beta^{\overline{\rm DR}}_{40001} = \frac{57}{256} - \frac{1}{256} n_f \,,$$

$$\beta^{\overline{\rm DR}}_{31100} = \frac{9}{512} n_f \,, \qquad \beta^{\overline{\rm DR}}_{31010} = -\frac{15}{64} n_f \,, \qquad \beta^{\overline{\rm DR}}_{31001} = -\frac{3}{256} n_f \,,$$

$$\beta^{\overline{\rm DR}}_{30200} = \frac{2223}{2048} \,, \qquad \beta^{\overline{\rm DR}}_{30020} = -\frac{855}{64} \,, \qquad \beta^{\overline{\rm DR}}_{30002} = \frac{441}{256} \,, \qquad \beta^{\overline{\rm DR}}_{30110} = \frac{45}{128} \,,$$

$$\beta^{\overline{\rm DR}}_{30101} = -\frac{801}{512} \,, \qquad \beta^{\overline{\rm DR}}_{30011} = -\frac{45}{64} \,, \qquad \beta^{\overline{\rm DR}}_{22100} = \frac{21}{128} n_f \,, \qquad \beta^{\overline{\rm DR}}_{22010} = -\frac{35}{192} n_f \,,$$

$$\beta^{\overline{\rm DR}}_{22001} = -\frac{7}{64} n_f \,, \qquad \beta^{\overline{\rm DR}}_{21200} = -\frac{9}{64} n_f \,, \qquad \beta^{\overline{\rm DR}}_{21020} = \frac{5}{4} n_f \,, \qquad \beta^{\overline{\rm DR}}_{21002} = -\frac{7}{32} n_f \,,$$

$$\beta^{\overline{\rm DR}}_{21101} = \frac{3}{16} n_f \,, \qquad \beta^{\overline{\rm DR}}_{20300} = -\frac{297}{1024} \,, \qquad \beta^{\overline{\rm DR}}_{20030} = 20 \,, \qquad \beta^{\overline{\rm DR}}_{20003} = -\frac{49}{128} \,,$$

$$\beta^{\overline{\rm DR}}_{20210} = -\frac{135}{128} \,, \qquad \beta^{\overline{\rm DR}}_{20201} = \frac{297}{512} \,, \qquad \beta^{\overline{\rm DR}}_{20120} = -\frac{45}{32} \,, \qquad \beta^{\overline{\rm DR}}_{20021} = \frac{105}{32} \,,$$

$$\beta^{\overline{\rm DR}}_{20102} = \frac{63}{128} \,, \qquad \beta^{\overline{\rm DR}}_{20012} = -\frac{105}{32} \,, \qquad \beta^{\overline{\rm DR}}_{20111} = \frac{45}{16}$$

- Susy limit: agreement with [I. Jack, D.R.T. Jones and A. Pickering '98]

$$\beta_s^{\rm SYM} = -\left(\frac{\alpha_s}{\pi}\right)^2 \left[\frac{3}{4}C_A + \frac{3}{8}C_A^2\frac{\alpha_s}{\pi} + \frac{21}{64}C_A^3\left(\frac{\alpha_s}{\pi}\right)^2 + \frac{51}{128}C_A^4\left(\frac{\alpha_s}{\pi}\right)^3\right] + \mathcal{O}(\alpha_s^6)$$

- massless QCD $\rightarrow$ Susy Yang-Mills theory:
  - adjust the colour factors: $C_A = C_F = 2T$, $n_f = 1$
  - set $\alpha_s^{\overline{\rm DR}} = \alpha_e = \eta_1$ and $\eta_2 = \eta_3 = 0$
- 3-loop β-function of the evanescent Yukawa coupling:

$$\beta_e^{\rm SYM} = \beta_s^{\rm SYM} + \mathcal{O}(\alpha_s^5)$$

  - disagreement with [L.V. Avdeev '82]
  - Susy preserved through 3-loops

3-loop $\gamma_m^{\overline{\mathrm{DR}}}$-function

- Explicit computation

$$\gamma_m^{\overline{\mathrm{DR}}} = -\pi\beta_s^{\overline{\mathrm{DR}}}\frac{\partial \ln Z_m^{\overline{\mathrm{DR}}}}{\partial \alpha_s^{\overline{\mathrm{DR}}}} - \pi\beta_e\frac{\partial \ln Z_m^{\overline{\mathrm{DR}}}}{\partial \alpha_e} - \pi\sum_r \beta_{\eta_r}\frac{\partial \ln Z_m^{\overline{\mathrm{DR}}}}{\partial \eta_r}$$

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- 2-loop result for $\alpha_e = \alpha_s^{\overline{\rm DR}}$ agrees with [L. Avdeev & M. Kalmykov'97]

$$\begin{aligned}\gamma_m^{\overline{\rm DR}}(\alpha_s^{\overline{\rm DR}},\alpha_e) &= 1 - \frac{\alpha_s^{\overline{\rm DR}}}{\pi}\frac{3}{4}C_F \\ &- \left[\frac{\alpha_s^{\overline{\rm DR}}}{\pi}\right]^2\left[\frac{3}{32}C_F^2 + \frac{91}{96}C_AC_F - \frac{10}{48}C_FTn_f\right] \\ &+ \frac{\alpha_s^{\overline{\rm DR}}}{\pi}\frac{\alpha_e}{\pi}\frac{3}{8}C_F^2 - \left[\frac{\alpha_e}{\pi}\right]^2\left[\frac{1}{4}C_F^2 - \frac{1}{8}C_AC_F + \frac{1}{8}C_FTn_f\right]\end{aligned}$$

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- 3-loop result:
  - 3-loop $Z_m^{\overline{\mathrm{DR}}}$ computed
  - 2-loop β_e computed
  - only tree-level η's contribute

3-loop $\gamma_m^{\overline{\rm DR}}$-function

- Explicit computation

$$\gamma_m^{\overline{\rm DR}} = -\pi\beta_s^{\overline{\rm DR}}\frac{\partial \ln Z_m^{\overline{\rm DR}}}{\partial\alpha_s^{\overline{\rm DR}}} - \pi\beta_e\frac{\partial \ln Z_m^{\overline{\rm DR}}}{\partial\alpha_e} - \pi\sum_r \beta_{\eta_r}\frac{\partial \ln Z_m^{\overline{\rm DR}}}{\partial\eta_r}$$

- 3-loop result:
  - 3-loop $Z_m^{\overline{\rm DR}}$ computed
  - 2-loop β_e computed
  - only tree-level η's contribute

$$\gamma_m^{\overline{\rm DR}}(\alpha_s^{\overline{\rm DR}},\alpha_e,\{\eta_r\}) = -\sum_{i,j,k,l,m}\gamma_{ijklm}^{\overline{\rm DR}}\left(\frac{\alpha_s^{\overline{\rm DR}}}{\pi}\right)^i\left(\frac{\alpha_e}{\pi}\right)^j\left(\frac{\eta_1}{\pi}\right)^k\left(\frac{\eta_2}{\pi}\right)^l\left(\frac{\eta_3}{\pi}\right)^m$$

$$
\begin{aligned}
\gamma_{30}^{\overline{\rm DR}} &= \frac{129}{128}C_F^3 - \frac{133}{256}C_F^2C_A + \frac{10255}{6912}C_FC_A^2 + \frac{-23+24\zeta_3}{32}C_F^2Tn_f \\
&\quad - \left(\frac{281}{864} + \frac{3}{4}\zeta_3\right)C_AC_FTn_f - \frac{35}{432}C_FT^2n_f^2\,, \\
\gamma_{21}^{\overline{\rm DR}} &= -\frac{27}{64}C_F^3 - \frac{21}{32}C_F^2C_A - \frac{15}{256}C_FC_A^2 + \frac{9}{32}C_F^2Tn_f\,, \\
\gamma_{12}^{\overline{\rm DR}} &= \frac{9}{8}C_F^3 - \frac{21}{32}C_F^2C_A + \frac{3}{64}C_FC_A^2 + \frac{3}{64}C_AC_FTn_f + \frac{3}{8}C_F^2Tn_f\,, \\
\gamma_{03}^{\overline{\rm DR}} &= -\frac{3}{8}C_F^3 + \frac{3}{8}C_F^2C_A - \frac{3}{32}C_FC_A^2 + \frac{1}{8}C_AC_FTn_f - \frac{5}{16}C_F^2Tn_f - \frac{1}{32}C_FT^2n_f^2\,, \\
\gamma_{02100}^{\overline{\rm DR}} &= \frac{3}{8}\,, \qquad \gamma_{02010}^{\overline{\rm DR}} = -\frac{5}{12}\,, \qquad \gamma_{02001}^{\overline{\rm DR}} = -\frac{1}{4}\,, \qquad \gamma_{01200}^{\overline{\rm DR}} = -\frac{9}{64}\,, \\
\gamma_{01020}^{\overline{\rm DR}} &= \frac{5}{4}\,, \qquad \gamma_{01101}^{\overline{\rm DR}} = \frac{3}{16}\,, \qquad \gamma_{01002}^{\overline{\rm DR}} = -\frac{7}{32}
\end{aligned}
$$

3-loop $\gamma_m^{\overline{\rm DR}}$-function (2)

- Indirect computation

$$\gamma_m^{\overline{\rm DR}} = \gamma_m^{\overline{\rm MS}} \frac{\partial \ln m^{\overline{\rm DR}}}{\partial \ln m^{\overline{\rm MS}}} + \frac{\pi \beta_s^{\overline{\rm MS}}}{m^{\overline{\rm DR}}} \frac{\partial m^{\overline{\rm DR}}}{\partial \alpha_s^{\overline{\rm MS}}} + \frac{\pi \beta_e}{m^{\overline{\rm DR}}} \frac{\partial m^{\overline{\rm DR}}}{\partial \alpha_e} + \ldots$$

  - 3-loop $\gamma_m^{\overline{\rm MS}}$ known
  - 2-loop β_e computed
  - 2-loop relation between $m^{\overline{\rm DR}}$ and $m^{\overline{\rm MS}}$ computed

$$\frac{m^{\overline{\rm DR}}}{m^{\overline{\rm MS}}} = 1 - \frac{\alpha_e}{\pi}\frac{1}{4}C_F - \frac{\alpha_s^{\overline{\rm MS}}}{\pi}\frac{\alpha_e}{\pi}\left[\frac{1}{4}C_F^2 + \frac{3}{32}C_A C_F\right] + \left[\frac{\alpha_s^{\overline{\rm MS}}}{\pi}\right]^2 \frac{11}{192} C_A C_F + \left[\frac{\alpha_e}{\pi}\right]^2 \left[\frac{3}{32}C_F^2 + \frac{1}{32}C_F T n_f\right]$$

- Equivalence of **DRED** and **DREG** at 3-loop order

4-loop $\gamma_m^{\overline{\rm MS}}$-function

- Indirect computation

$$\gamma_m^{\overline{\rm DR}} = \gamma_m^{\overline{\rm MS}} \frac{\partial \ln m^{\overline{\rm DR}}}{\partial \ln m^{\overline{\rm MS}}} + \frac{\pi \beta_s^{\overline{\rm MS}}}{m^{\overline{\rm DR}}} \frac{\partial m^{\overline{\rm DR}}}{\partial \alpha_s^{\overline{\rm MS}}} + \frac{\pi \beta_e}{m^{\overline{\rm DR}}} \frac{\partial m^{\overline{\rm DR}}}{\partial \alpha_e} + \sum_r \frac{\pi \beta_{\eta_r}}{m^{\overline{\rm DR}}} \frac{\partial m^{\overline{\rm DR}}}{\partial \eta_r}$$

  - 4-loop $\gamma_m^{\overline{\rm MS}}$ known [K. Chetyrkin '97, J. Vermaseren et al '97]
  - 3-loop β_e computed ($\simeq 10.000$ diagrams)
  - 3-loop relation between $m^{\overline{\rm DR}}$ and $m^{\overline{\rm MS}}$ computed

4-loop $\gamma_m^{\overline{\mathrm{MS}}}$-function

- Indirect computation

$$\gamma_m^{\overline{\mathrm{DR}}} = \gamma_m^{\overline{\mathrm{MS}}} \frac{\partial \ln m^{\overline{\mathrm{DR}}}}{\partial \ln m^{\overline{\mathrm{MS}}}} + \frac{\pi \beta_s^{\overline{\mathrm{MS}}}}{m^{\overline{\mathrm{DR}}}} \frac{\partial m^{\overline{\mathrm{DR}}}}{\partial \alpha_s^{\overline{\mathrm{MS}}}} + \frac{\pi \beta_e}{m^{\overline{\mathrm{DR}}}} \frac{\partial m^{\overline{\mathrm{DR}}}}{\partial \alpha_e} + \sum_r \frac{\pi \beta_{\eta_r}}{m^{\overline{\mathrm{DR}}}} \frac{\partial m^{\overline{\mathrm{DR}}}}{\partial \eta_r}$$

  - 4-loop $\gamma_m^{\overline{\mathrm{MS}}}$ known [K. Chetyrkin '97, J. Vermaseren et al '97]
  - 3-loop β_e computed ($\simeq 10.000$ diagrams)
  - 3-loop relation between $m^{\overline{\mathrm{DR}}}$ and $m^{\overline{\mathrm{MS}}}$ computed

$$\begin{aligned}
\delta_m^{(3)} &= \left(\frac{\alpha_s^{\overline{\mathrm{MS}}}}{\pi}\right)^3 \left(\frac{2207}{864} + \frac{19}{648} n_f\right) - \frac{\left(\alpha_s^{\overline{\mathrm{MS}}}\right)^2 \alpha_e}{\pi^3} \left(\frac{62815}{20736} + \frac{253}{1728} n_f - \frac{25}{72} n_f^2\right) \\
&+ \frac{\alpha_s^{\overline{\mathrm{MS}}} \alpha_e^2}{\pi^3} \left[\frac{1973}{2592} - \frac{5}{36}\zeta_3 + \left(\frac{103}{1728} + \frac{5}{36}\zeta_3\right) n_f\right] - \frac{\alpha_e^2 \eta_2}{\pi^3} \frac{5}{24} \\
&- \left(\frac{\alpha_e}{\pi}\right)^3 \left(\frac{7}{144} + \frac{5}{216}\zeta_3 + \frac{31}{576} n_f - \frac{5}{576} n_f^2\right) - \frac{\alpha_e}{\pi^3} \left(\eta_1^2 \frac{9}{256} - \eta_2^2 \frac{15}{16} - \eta_1 \eta_3 \frac{3}{64} + \eta_3^2 \frac{7}{128}\right)
\end{aligned}$$

4-loop $\gamma_m^{\overline{\mathrm{MS}}}$-function

- Indirect computation

$$\gamma_m^{\overline{\mathrm{DR}}} = \gamma_m^{\overline{\mathrm{MS}}} \frac{\partial \ln m^{\overline{\mathrm{DR}}}}{\partial \ln m^{\overline{\mathrm{MS}}}} + \frac{\pi \beta_s^{\overline{\mathrm{MS}}}}{m^{\overline{\mathrm{DR}}}} \frac{\partial m^{\overline{\mathrm{DR}}}}{\partial \alpha_s^{\overline{\mathrm{MS}}}} + \frac{\pi \beta_e}{m^{\overline{\mathrm{DR}}}} \frac{\partial m^{\overline{\mathrm{DR}}}}{\partial \alpha_e} + \sum_r \frac{\pi \beta_{\eta_r}}{m^{\overline{\mathrm{DR}}}} \frac{\partial m^{\overline{\mathrm{DR}}}}{\partial \eta_r}$$

  - 4-loop $\gamma_m^{\overline{\mathrm{MS}}}$ known [K. Chetyrkin ’97, J. Vermaseren et al ’97]
  - 3-loop β_e computed ($\simeq 10.000$ diagrams)
  - 3-loop relation between $m^{\overline{\mathrm{DR}}}$ and $m^{\overline{\mathrm{MS}}}$ computed

- agreement for the Susy limit with [I. Jack & D.R.T. Jones ’97]

$$\gamma_m^{\mathrm{SYM}} = \pi \alpha_s \frac{\mathrm{d}}{\mathrm{d}\alpha_\mathrm{s}} \left[\frac{\beta_s^{\mathrm{SYM}}}{\alpha_s} \right]$$

$$\gamma^{\overline{\rm DR}}_{40000} = \gamma_3 - \frac{18763}{2304} + \left(\frac{1}{6} + \frac{5}{8}\zeta_3\right)n_f + \frac{29}{5184}n_f^2 ,$$

$$\gamma^{\overline{\rm DR}}_{31000} = -\frac{147659}{4608} + \frac{125}{48}\zeta_3 + \left(\frac{58253}{31104} + \frac{95}{216}\zeta_3\right)n_f + \frac{407}{7776}n_f^2 ,$$

$$\gamma^{\overline{\rm DR}}_{22000} = -\frac{134147}{62208} - \frac{281}{432}\zeta_3 + \left(\frac{336497}{124416} + \frac{49}{432}\zeta_3\right)n_f - \left(\frac{181}{10368} + \frac{5}{216}\zeta_3\right)n_f^2 ,$$

$$\gamma^{\overline{\rm DR}}_{13000} = -\frac{595}{7776} - \frac{25}{108}\zeta_3 - \left(\frac{1163}{10368} - \frac{5}{27}\zeta_3\right)n_f - \left(\frac{145}{3456} + \frac{5}{72}\zeta_3\right)n_f^2 ,$$

$$\gamma^{\overline{\rm DR}}_{04000} = \frac{191}{2592} + \frac{67}{108}\zeta_3 + \left(\frac{301}{1728} - \frac{1}{24}\zeta_3\right)n_f + \frac{5}{384}n_f^2 - \frac{5}{768}n_f^3 ,$$

$$\gamma^{\overline{\rm DR}}_{30100} = \frac{9}{256} , \quad \gamma^{\overline{\rm DR}}_{30010} = -\frac{15}{32} , \quad \gamma^{\overline{\rm DR}}_{30001} = -\frac{3}{128} , \quad \gamma^{\overline{\rm DR}}_{21100} = \frac{201}{512} ,$$

$$\gamma^{\overline{\rm DR}}_{21010} = -\frac{85}{64} , \quad \gamma^{\overline{\rm DR}}_{21001} = -\frac{107}{256} , \quad \gamma^{\overline{\rm DR}}_{20200} = -\frac{27}{256} , \quad \gamma^{\overline{\rm DR}}_{20020} = \frac{15}{16} ,$$

$$\gamma^{\overline{\rm DR}}_{20002} = -\frac{21}{128} , \quad \gamma^{\overline{\rm DR}}_{20101} = \frac{9}{64} , \quad \gamma^{\overline{\rm DR}}_{12100} = \frac{351}{64} , \quad \gamma^{\overline{\rm DR}}_{12010} = -\frac{365}{96} ,$$

$$\gamma^{\overline{\rm DR}}_{12001} = -\frac{117}{32} , \quad \gamma^{\overline{\rm DR}}_{11200} = -\frac{1563}{512} , \quad \gamma^{\overline{\rm DR}}_{11020} = \frac{1645}{96} , \quad \gamma^{\overline{\rm DR}}_{11002} = -\frac{3647}{768} ,$$

$$\gamma^{\overline{\rm DR}}_{11101} = \frac{521}{128} , \quad \gamma^{\overline{\rm DR}}_{03100} = -\frac{13}{64} - \frac{45}{64}n_f , \quad \gamma^{\overline{\rm DR}}_{03010} = \frac{55}{96}n_f ,$$

$$\gamma^{\overline{\rm DR}}_{03001} = \frac{13}{96} + \frac{15}{32}n_f , \quad \gamma^{\overline{\rm DR}}_{02200} = -\frac{223}{256} + \frac{153}{512}n_f , \quad \gamma^{\overline{\rm DR}}_{02020} = \frac{395}{144} - \frac{65}{32}n_f ,$$

$$\gamma^{\overline{\rm DR}}_{02002} = \frac{259}{1152} + \frac{119}{256}n_f , \quad \gamma^{\overline{\rm DR}}_{02110} = -\frac{155}{48} , \quad \gamma^{\overline{\rm DR}}_{02101} = \frac{233}{192} - \frac{51}{128}n_f ,$$

$$\gamma^{\overline{\rm DR}}_{02011} = \frac{545}{144} , \quad \gamma^{\overline{\rm DR}}_{01300} = \frac{333}{512} , \quad \gamma^{\overline{\rm DR}}_{01030} = -20 , \quad \gamma^{\overline{\rm DR}}_{01003} = -\frac{7}{192} ,$$

$$\gamma^{\overline{\rm DR}}_{01210} = \frac{105}{64} , \quad \gamma^{\overline{\rm DR}}_{01201} = -\frac{333}{256} , \quad \gamma^{\overline{\rm DR}}_{01120} = -\frac{5}{16} , \quad \gamma^{\overline{\rm DR}}_{01021} = \frac{35}{48} ,$$

$$\gamma^{\overline{\rm DR}}_{01102} = \frac{3}{64} , \quad \gamma^{\overline{\rm DR}}_{01012} = \frac{245}{48} , \quad \gamma^{\overline{\rm DR}}_{01111} = -\frac{35}{8}$$

Phenomenological analysis

- non-SUSY theories $\alpha_s^{\overline{\mathrm{DR}}} \neq \alpha_e$ essential
- Numerical example
  - SUSY theory : $\beta_s^{\overline{\mathrm{DR}}} = \beta_e$

    $\alpha_s^{\overline{\mathrm{DR}}}(\mu) = \alpha_e(\mu)$ at all scales.
  - Integrate out all SUSY particles at $\mu = M_Z$

    $\alpha_s^{\overline{\mathrm{DR}}}(M_Z) = \alpha_e(M_Z) = 0.120$
  - Evolve $\alpha_s^{\overline{\mathrm{DR}}}$ and α_e down to $\mu_b = 4.2\,$GeV

    $\alpha_s^{\overline{\mathrm{DR}}}(\mu_b) = 0.218$

    $\alpha_e(\mu_b) = 0.167$

    $m_b^{\overline{\mathrm{DR}}}(\mu_b) = 4.12$

  If $\alpha_s^{\overline{\mathrm{DR}}} = \alpha_e \quad \Rightarrow \quad \delta_m \simeq 30\,$MeV

Conclusions

- 4-loop QCD β-function and mass anomalous dimension γ_m computed within DRED
  - explicit calculation of β and γ_m to 3-loop order
  - 4-loop relation between DRED and DREG established
- Equivalence of DRED and DREG at 3-loop order
- Susy-YM: supersymmetry preserved through 3-loop order
- Careful treatment of the evanescent couplings phenomenologically important