Four-loop calculations for a precise charm- and bottom-quark mass determination

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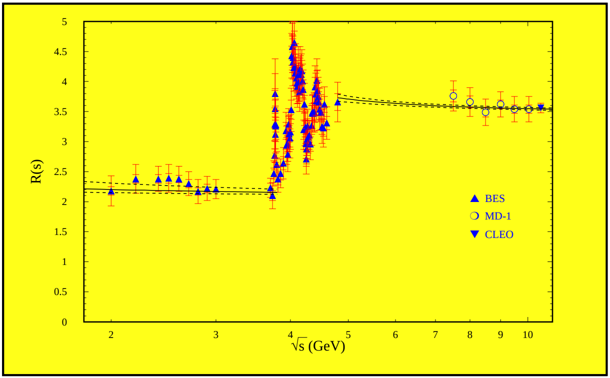
Introduction
Motivation

Precise determination of the charm- and bottom-quark mass is important:

- Quark masses are fundamental parameters of the standard modell
- Higgs-physics
- Flavour-physics
- Comparison with other methods: lattice calculations,...

Introduction
Experiment

$$R(s) = \frac{\sigma(e^+e^- \to \text{Hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)}$$



Introduction

Theory

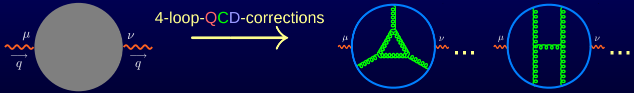
- Correlator of two currents:

$$\Pi^{\mu\nu}(q,j) = i \int dx\, e^{iqx} \langle 0|T j^{\mu}(x) j^{\nu}(0)|0\rangle$$

here: $j^{\mu}(x)$ electromagnetic heavy quark current

- Diagrammatically:

$$i\Pi^{\mu\nu}(q) =$$



- Relation to polarization function $\Pi(q^2)$:

$$\Pi^{\mu\nu}(q) = (-g^{\mu\nu}q^2 + q^{\mu}q^{\nu})\,\Pi(q^2)$$

Introduction

Relation: theory $\Longleftrightarrow$ experiment

- With the help of dispersion-relations:

$$\Pi(q^2) = \Pi(q^2=0) + \frac{q^2}{12\pi^2}\int ds \frac{R(s)}{s(s-q^2)}$$

exp. moments are related to derivatives of $\Pi(q^2)$ at $q^2 = 0$:

$$\frac{12\pi}{n!}\left(\frac{d}{dq^2}\right)^n \Pi(q^2)\Big|_{q^2=0} = \mathcal{M}_n^{\text{exp}} = \int \frac{ds}{s^{n+1}} R^{\text{exp}}(s)$$

- In terms of expansion coefficients:

$$\Pi(q^2) = \frac{3Q_f^2}{16\pi^2}\sum_n \overline{C}_n \left(\frac{q^2}{4m^2}\right)^n,$$

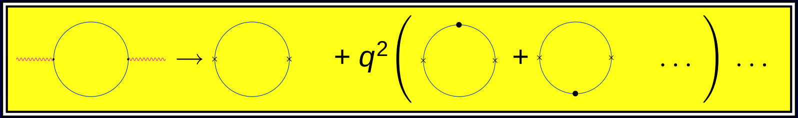
Q_f: charge of quark
$m = m(\mu)$: $\overline{\text{MS}}$ mass

SVZ: $\overline{C}_n$ can be calculated perturbatively

Introduction

Relation: theory $\Longleftrightarrow$ experiment

- Expansion diagrammatically:



- First and higher derivatives of $\Pi(q^2)$ allow direct determination of the $\overline{\text{MS}}$ charm- and bottom-quark mass:

$$m(\mu) = \frac{1}{2}\left(Q_f^2 \frac{9}{4} \frac{\overline{C}_n}{\mathcal{M}_n^{\text{exp}}}\right)^{1/(2n)}$$

← Theory

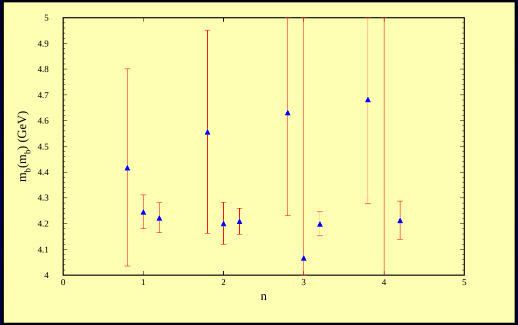
← Experiment

c-quarks: Novikov, et al. '78; b-quarks: Reinders, et al. '85

$\overline{C}_n$ depend on the quark mass through $\log(m(\mu)^2/\mu^2)$

Motivation

at 3-loop



J.H. Kühn, M. Steinhauser '01

$\overline{\text{MS}}$ – mass:

$m_b(m_b) = 4.191(51)$ GeV

. . . analog for charm-quarks:

$m_c(m_c) = 1.304(27)$ GeV

Present status:

Order α_s^2 coefficients $\overline{C}_n$ up to n=8 K.G. Chetyrkin, J.H. Kühn, M. Steinhauser '96

recently up to n=30 R. Boughezal, M. Czakon,T. Schutzmeier '06

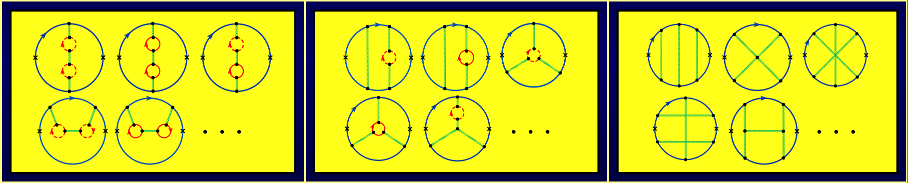
Order α_s^3 coefficients $\overline{C}_n$ n=0,1 Chetyrkin, Kühn, C.S. '06 ⟵ in this talk

confirmed by R. Boughezal, M. Czakon,T. Schutzmeier '06

Strategy

Decomposition with respect to the number of inserted closed Fermion-lines

n_f^2-contributions $\quad$ n_f^1-contributions $\quad$ n_f^0-contributions



: heavy quarks, : light quarks,
n_f: number of active quarks

$\Longrightarrow$ About 700 Feynman-diagrams

Strategy

Integration-by-parts:

K.G. Chetyrkin, F.V. Tkachov

$$0 = \int [d^D k_1] \ldots [d^D k_4] \quad \partial_{(k_j)_\mu} \left(k_l^\mu \; I_{\alpha\beta} \right) \;, \qquad j, l = 1, \ldots, \text{loops}=4$$

$I_{\alpha\beta}$: Generic integrand with propagator powers $\alpha = \{\alpha_1, \ldots\}$

and scalar-product powers $\beta = \{\beta_1, \ldots\}$

Laporta-Algorithm:

S. Laporta, E. Remiddi

Idea: – IBP-identities for explicit numerical values of α, β

– Introduction of an order among the integrals

– Solving a linear system of equations

Problem: Dramatic growth of number of equations

Here: >31 million IBP-identities generated and solved

$\rightsquigarrow$ Integral-tables with solutions for around

5 million integrals, expressed through 13 masters

Methods

Important: Consider all symmetries of diagrams

$\rightsquigarrow$ Smaller number of IBP-equations,

$\rightsquigarrow$ Keep size of integral-tables under control

Automation:

Generation & solution of the system of linear equations with:

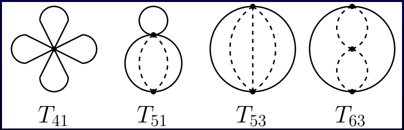
– Implementation based on `FORM3` J.A.M. Vermaseren

– Simplification of rational functions in *d* by `FERMAT` R.H. Lewis

– with the use of `GateToFermat` M. Tentyoukov, J.A.M. Vermaseren

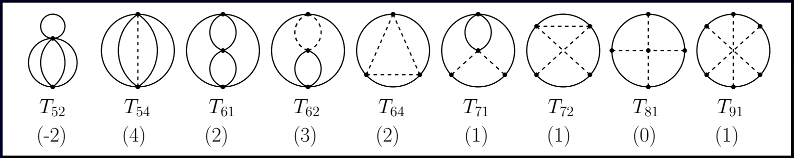
Remains: Solution of 13 master integrals(MI):

four are simple ...



Methods

Master integrals



Solution with high precision numerics Y. Schröder, A. Vuorinen
with difference equation method S. Laporta
other contributions: Broadhurst; Laporta; Kniehl, Kotikov; Schröder, Steinhauser

Diffi culty:

- Solving IBP-identities $\Rightarrow$ Division by $(d - 4)$ can appear
  $\rightsquigarrow$ "spurious" poles
  $\rightsquigarrow$ Master integrals with spurious poles as coeffi cient need to be known in higher order in ε

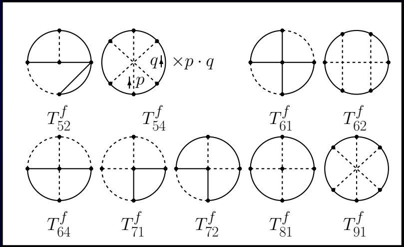
Method

ε-fi nite basis K.G. Chetyrkin, M. Faisst, C.S., M. Tentyukov

But: Choice of master integrals is not unique

$\rightsquigarrow$ Select a new basis with ε-finite coefficients

This ε-finite basis can be found in the set of initial integrals F_i



Prescription:

1.) Select F_i having the $1/\varepsilon^{n_{max}}$-pole with highest power n_{max}

2.) Solve for master integral $T_{j,n_{max}}$ and replace $T_{j,n_{max}} \rightarrow F_i \equiv T^f_{j,n_{max}}$

3.) Repeat 1.) and 2.) until no spurious poles survive

Result

K.G. Chetyrkin, J. H. Kühn, C.S.

$$\Pi(q^2) = \frac{3}{16\pi^2}\Bigg\{ \left(\frac{\alpha_s}{\pi}\right) 1.4444 + \left(\frac{\alpha_s}{\pi}\right)^2 \left(1.5863 + 0.1387 n_h + 0.3714 n_l\right)$$

$$+ \left(\frac{\alpha_s}{\pi}\right)^3 \Big(0.0257 n_l^2 - 0.0309 n_h^2 + 0.0252 n_h n_l$$

$$-1.2112 n_l - 3.3426 n_h + 1.4186\Big)$$

$$+ \left(\frac{q^2}{4\overline{m}^2}\right)\Big[1.0667$$

$$+ \left(\frac{\alpha_s}{\pi}\right) 2.5547 + \left(\frac{\alpha_s}{\pi}\right)^2 \left(0.2461 + 0.2637 n_h + 0.6623 n_l\right)$$

$$+ \left(\frac{\alpha_s}{\pi}\right)^3 \Big(0.0961 n_l^2 + 0.0130 n_h^2 + 0.1658 n_h n_l$$

$$-2.9605 n_l - 6.4188 n_h + 8.2846\Big)\Big] + \ldots \Bigg\}$$

evaluated for scale $\mu = \overline{m}$, confi rmed by R. Boughezal, M. Czakon, T. Schutzmeier

...for charm-quarks

$$
\begin{aligned}
\overline{C}_n = \overline{C}_n^{(0)} \quad &+ \quad \left(\frac{\alpha_s}{\pi}\right)\left(\overline{C}_n^{(10)} + \overline{C}_n^{(11)} l_m\right) \\
&+ \quad \left(\frac{\alpha_s}{\pi}\right)^2 \left(\overline{C}_n^{(20)} + \overline{C}_n^{(21)} l_{m_c} + \overline{C}_n^{(22)} l_{m_c}^2\right) \\
&+ \quad \left(\frac{\alpha_s}{\pi}\right)^3 \left(\overline{C}_n^{(30)} + \overline{C}_n^{(31)} l_{m_c} + \overline{C}_n^{(32)} l_{m_c}^2 + \overline{C}_n^{(33)} l_{m_c}^3\right) \\
&+ \quad \ldots, \text{with } l_{m_c} = \log(m_c^2/\mu^2)
\end{aligned}
$$

	1-loop	2-loop		3-loop			4-loop			
n	$\overline{C}_n^{(0)}$	$\overline{C}_n^{(10)}$	$\overline{C}_n^{(11)}$	$\overline{C}_n^{(20)}$	$\overline{C}_n^{(21)}$	$\overline{C}_n^{(22)}$	$\overline{C}_n^{(30)}$	$\overline{C}_n^{(31)}$	$\overline{C}_n^{(32)}$	$\overline{C}_n^{(33)}$
1	1.0667	2.5547	2.1333	2.4967	3.3130	-0.0889	-7.7624	-0.0599	1.5851	-0.0543

$$
m(\mu) = \frac{1}{2}\left(Q_c^2 \frac{9}{4} \frac{\overline{C}_n}{\mathcal{M}_n^{\text{exp}}}\right)^{1/(2n)}
$$

Implications

Quark masses

New multi-loop results from pQCD and
improved experimental data (update on $\Gamma_e(J/\Psi), \Psi'$)
lead to significantly reduced errors:

- Preliminary result for charm-quark:

  n=1 N^3LO:

  $$m_c(3GeV) = 991 \pm 11(exp) \pm 8(\alpha_s) \pm 0.3(scale) \pm 1(np)$$
  $$\Rightarrow m_c(m_c) = 1290 \pm 13$$

- Result for bottom-quark: work in progress...

Summary & Conclusions

- Calculation of higher Taylor-coefficients of the polarization function allows a precise determination of the charm- and bottom-quark mass
- Reduction to master integrals of the first two Taylor coefficients in 4-loop order in perturbative QCD is completed; Master integrals have been calculated
- New multi-loop results from pQCD + improved data lead to significantly reduced errors
  preliminary value for $m_c(m_c) = 1290 \pm 13$ MeV
- Outlook
  - Similar analysis for bottom-quark mass is on the way...