New results for four-loop tadpoles and $\mathcal{O}(G_F m_t^2 \alpha_s^3)$ corrections to the ρ-parameter

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Introduction and motivation

- ρ measures the relative strength of charged and neutral currents

$$\rho = \frac{M_W^2}{c_W^2\, M_Z^2} = 1 \quad \text{in SM at tree level}$$

Higher order corrections modify it into:

$$\rho = \frac{1}{1 - \Delta\rho} \qquad \text{where} \qquad \boxed{\Delta\rho = \frac{\Sigma_Z^T(0)}{M_Z^2} - \frac{\Sigma_W^T(0)}{M_W^2}}$$

- $\Delta\rho^{(1)} \sim M_t^2 \longrightarrow$ a sensitivity that was used in the indirect determination of M_t
  - remarkable agreement between theoretical prediction and experimentally measured M_t
- bounds on M_H depend critically on the knowledge of M_t

Introduction and motivation

- leading universal corrections to ρ give the main contribution to M_W and $\sin^2\theta^{\text{lept}}_{\text{eff}}$

$$\Delta M_W = \frac{M_W}{2}\,\frac{c_W^2}{c_W^2 - s_W^2}\,\Delta\rho \quad \text{and} \quad \Delta\sin^2\theta^{\text{lept}}_{\text{eff}} = -\,\frac{c_W^2\, s_W^2}{c_W^2 - s_W^2}\,\Delta\rho$$

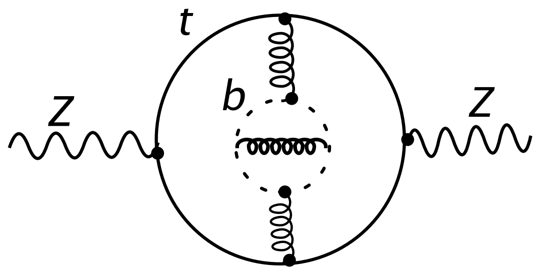
  - both are used in the indirect determination of m_H
  - current exp. uncertainty $\delta M_W = 34\text{MeV}$, $\delta\sin^2\theta^{\text{lept}}_{\text{eff}} = 1.7 \times 10^{-4}$

    ILC $\longrightarrow$ $\delta M_W = 6\text{MeV}$, $\delta\sin^2\theta^{\text{lept}}_{\text{eff}} = 1.3 \times 10^{-5}$
  - On-shell $\mathcal{O}(G_F\, M_t^2\, \alpha_s^2) \longrightarrow \Delta M_W = -10\text{MeV}$, $\Delta\sin^2\theta^{\text{lept}}_{\text{eff}} = 5 \times 10^{-5}$

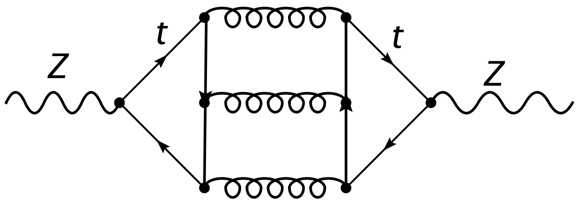
    $\Longrightarrow \mathcal{O}(G_F\, M_t^2\, \alpha_s^3)$ could improve the theoretical precision . . .

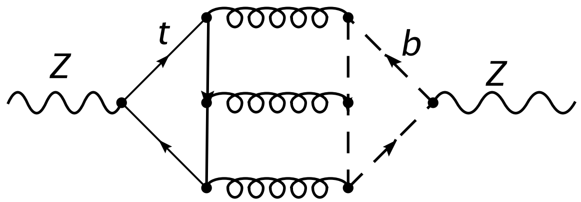
Calculation

- Consider the contribution of $\begin{pmatrix} t \\ b \end{pmatrix}$ for large $M_t \longrightarrow M_b = 0$
- Only the axial current correlator contributes to $\Sigma_Z^T(0)$
  - non-singlet Feynman diagrams: the external current couples to the same closed fermion loop



  - singlet Feynman diagrams: external Z couple to two different closed fermion loops





Calculation

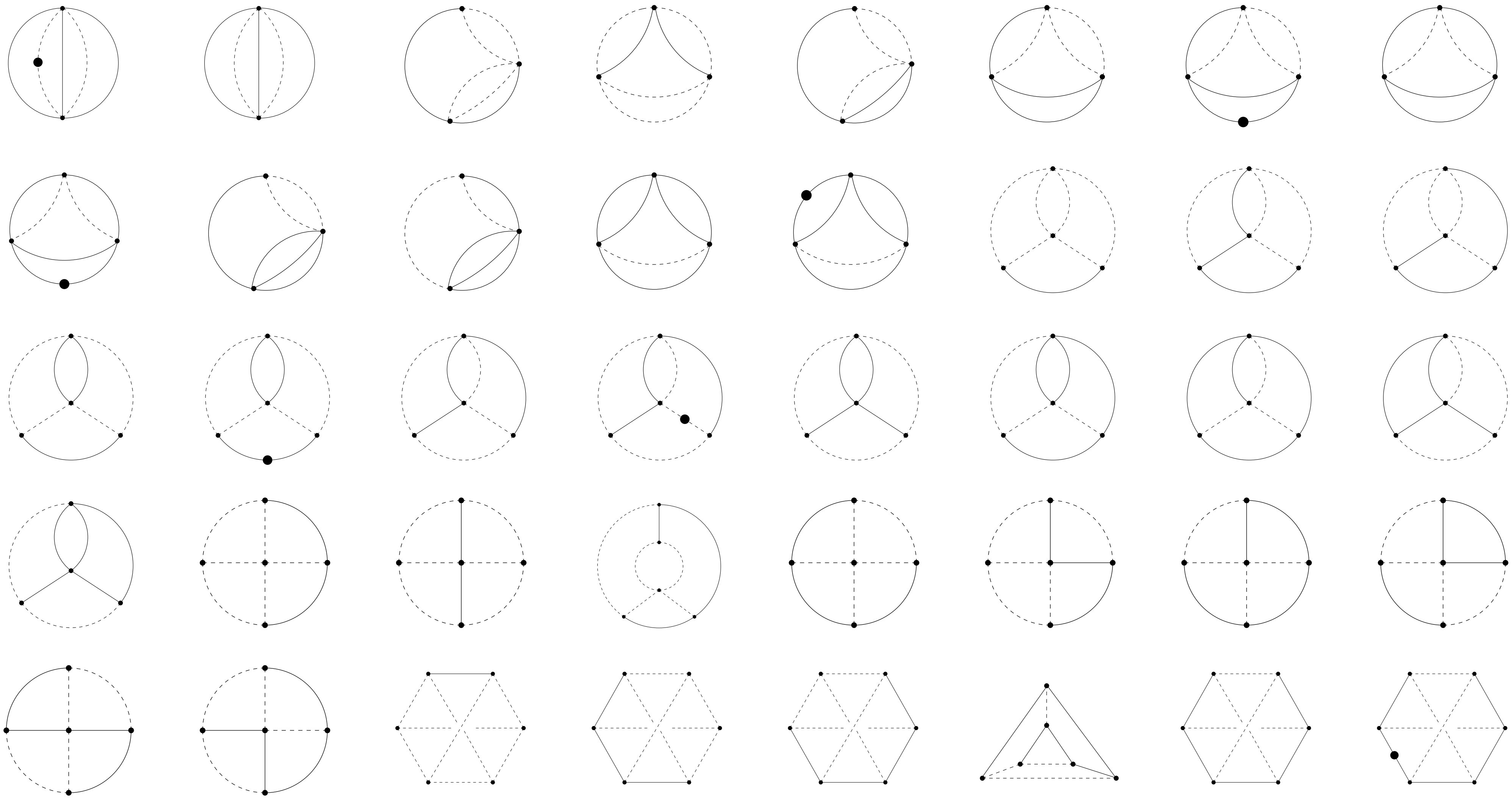
- Singlets
  - appear for the first time at the 3-loop level
  - require a special treatment because of their relation to the axial anomaly
  - form a finite and gauge independent subset
  - at the 3-loop level, in $\overline{MS}$, they are 20 times larger than the non-singlet piece
  - $\Delta\rho^{(4)}_{singlet}$ `Schröder & Steinhauser 05`
    - needs 13 master integrals
    - "numerical size of the result is surprisingly small" ...

      $\Longrightarrow$ definite conclusions require the non-singlet contribution to ρ

Calculation

- A complete evaluation of $\mathcal{O}(G_F\, M_t^2\, \alpha_s^3)$ to $\boldsymbol{\rho}$ requires:
  - non-singlet $\Sigma_Z^T(0)$
  - $\Sigma_W^T(0)$
- it goes in steps
  - generation of diagrams: *DiaGen* `M.Czakon`
  - algebraic reduction to master integrals: *IdSolver* `M.Czakon`
  - calculation of the masters
    - $\Sigma_Z^{T,non-singlet}(0)$ needs 13 masters $\longrightarrow$ already available analytically and numerically
    - $\Sigma_W^T(0)$ needs 63 masters
      - 13 are available
      - 10 factorisable / calculable in terms of Gamma functs
      - 40 non-trivial new 4-loop single scale tadpoles to calculate

The 40 new masters



Linear difference equations for the masters (`Laporta 01`)

- raise arbitrarily a massive propagator to a symbolic power x

$$B_{D_1}(x) = \int \frac{d^d k_1\, d^d k_2 \ldots d^d k_{N_k}}{D_1^x\, D_2 \ldots D_{N_D}} \qquad \text{with} \qquad D_i = k_i^2 + m_i^2$$

- use IBP identities to get $\boxed{\sum_{j=0}^{R} P_j(x)\, B_{D_1}(x+j) = F(x)}$ difference eqt of order R
  - set of master functs satisfies a triangular system of diff. eqts, the simplest master satisfies a homogeneous diff. eqt.
  - choices based on non-symmetric propagators lead to different difference eqts. Condition: $B_{D_i}(1) = B \Longrightarrow$ consistency check
- general solution for integer x $\boxed{B(x) = \sum_{j=1}^{R} a_j\, B_j^{(HO)}(x) + B^{(NH)}(x)}$

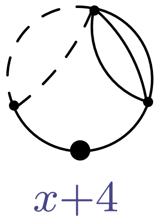
Linear difference equations for the masters (`Laporta 01`)

- solve by making the ansatz:

$$B^{(\alpha)}(x) = (\mu^{(\alpha)})^x \underbrace{\sum_{s=0}^{\infty} a_s^{(\alpha)} \frac{\Gamma(x+1)}{\Gamma(x+1-k^{(\alpha)}+s)}}_{\text{factorial series}} = (\mu^{(\alpha)})^x V^{(\alpha)}(x)$$

- find the behaviour of the integral for large x then use the boundary condition to get $\boldsymbol{a_0^H}$
- sum the factorial series at a large x then use the difference equation for $B(x)$ to compute $B(1)$
- instability of the recurrence relation limits our choice of x due to loss of precision

An example



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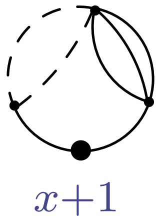
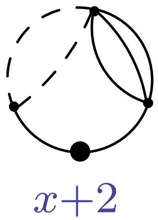


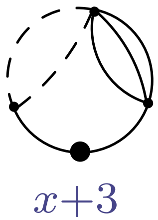
$$\frac{((x^2 + (7\,\varepsilon - 3)\,x + 12\,\varepsilon^2 - 10\,\varepsilon + 2)}{(8\,x^2 + (16\,\varepsilon + 32)\,x + 48\,\varepsilon + 24))}$$

$-$



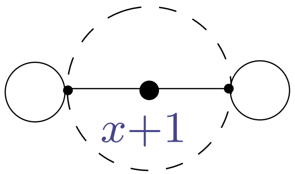
$$\frac{((-6\,x^2 + (-29\,\varepsilon - 2)\,x - 34\,\varepsilon^2 - 8\,\varepsilon + 2)}{(8\,x^2 + (16\,\varepsilon + 32)\,x + 48\,\varepsilon + 24))}$$

$-$



$$\frac{((-15\,x^2 + (-52\,\varepsilon - 31)\,x - 28\,\varepsilon^2 - 82\,\varepsilon - 6)}{(8\,x^2 + (16\,\varepsilon + 32)\,x + 48\,\varepsilon + 24))}$$

$=$

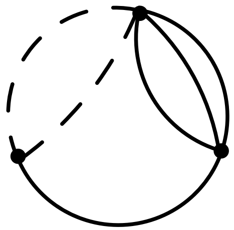


$$\frac{(((-3\,\varepsilon^3 + 9\,\varepsilon^2 - 9\,\varepsilon + 3)\,x - 6\,\varepsilon^4 + 21\,\varepsilon^3 - 27\,\varepsilon^2 + 15\,\varepsilon - 3)}{(2x^4 + (4\varepsilon + 14)x^3 + (24\varepsilon + 34)x^2 + (44\varepsilon + 34)x + 24\varepsilon + 12))}$$

- a difference equation of order 3

An example

- the full result is



$$
\begin{aligned}
= \; & + 0.875000000000000000000000000000 \; \varepsilon^{-4} \\
& + 5.31250000000000000000000000000 \; \varepsilon^{-3} \\
& + 24.1272857172567359185353492916 \; \varepsilon^{-2} \\
& + 71.5869874700921357789204087564 \; \varepsilon^{-1} \\
& + 197.558371935133443514418819015 \\
& + 6.41187248372530806306191468163 \; \varepsilon \\
& - 1028.33128833387533834458667353 \; \varepsilon^{2} \\
& - 16249.8777552641957747492917792 \; \varepsilon^{3} \\
& - 65696.8989733875039196986140933 \; \varepsilon^{4}
\end{aligned}
$$

An example: convergence

x_{max}	s_{max}	ε^{-4}	ε^{-3}	ε^{-2}
10	500	0.87497975239619656213	5.31260791405574477774	24.1271074099526698940
10	1100	0.87499959532770610508	5.31250338417602436777	24.127273308267639324
30	500	0.87499999999999911820	5.31250000000000104701	24.127285717256695507
30	1100	0.87500000000000000000	5.31250000000000000000	24.127285717256735919
90	500	0.87500000000000000000	5.31250000000000000001	24.127285717256735918
90	1100	0.87500000000000000000	5.31250000000000000000	24.127285717256735919

- stability of the digits requires a sufficiently high x and s

Implementation, running parameters and time

- Derivation of difference equations: $C^{++}/\texttt{Form}$
  - highest order 4
  - longest running time 52 hours on 1 CPU (a non-planar 9-liner)
- numerical solution of the difference eqts: `Mathematica`
  - longest running time 40 hours on 1 CPU (a non-planar 9-liner)
- initial depth of epsilon expansion is 12, many terms cancel when using the recurrence relation for $B(x)$
- all the masters but three 9-liners were solved for

$$\left.\begin{array}{rcl} S_{max} & = & 900-1100 \\ X_{max} & = & 90 \end{array}\right\} \text{Provided at least } 30 \text{ digits}$$

- for the particular three 9-liners, we have weak convergence

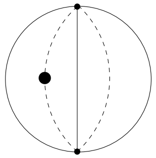
$$\left.\begin{array}{rcl} S_{max} & = & 750-900 \\ X_{max} & = & 55 \end{array}\right\} \text{Provided up to } 23 \text{ digits}$$

Checks

- checks on the master integrals:
  - Attaching x to different massive propagators leads to different master functs $B(x)$. For $x = 1$ they all coincide with the original master integral $\Longrightarrow$ an important internal check
  - stability of the numerical coefficients in the ε-expansion checked by varying S_{max}
  - Many masters were checked with sector decomposition method and Mellin-Barnes (MB-package `M.Czakon 05`)
- checks on the diagrams
  - cancellation of the first power of the gauge-fixing parameter ξ
  - agreement on the result of $\Delta\rho$ with the fully independent calculation of `Chetyrkin et al hep-ph/0605201`

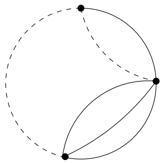
Results: few 4-loop masters

- All the 40 4-loop single scale tadpole masters have been calculated:
  - all the $5-$, 6- and 7-liners are provided up to ε^4 with 30 significant digits
  - all the 8- and 9-liners (but three) are given up to ε^3 with 30 significant digits



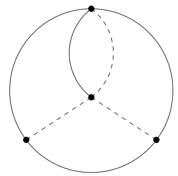
$$
\begin{aligned}
= \quad & -0.2500000000000000000000000000000\;\varepsilon^{-4} \\
& -1.12500000000000000000000000000\;\varepsilon^{-3} \\
& -3.32246703342411321823620758332\;\varepsilon^{-2} \\
& -6.69920730307326055744986988775\;\varepsilon^{-1} \\
& +30.0253835218317623192659472127 \\
& +254.111031964704093539375826207\;\varepsilon \\
& +1805.36593366425290012409359932\;\varepsilon^{2} \\
& +8898.24672362282959032538518673\;\varepsilon^{3} \\
& +43751.2551818001444717625115350\;\varepsilon^{4}
\end{aligned}
$$

Results: few 4-loop masters

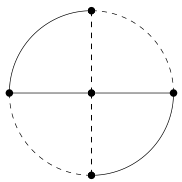


$$
\begin{aligned}
= \quad & + 0.625000000000000000000000000000 \; \varepsilon^{-4} \\
& + 4.125000000000000000000000000000 \; \varepsilon^{-3} \\
& + 20.669402017165956033632475041 6 \; \varepsilon^{-2} \\
& + 74.250354499477468533736621064 4 \; \varepsilon^{-1} \\
& + 147.807870353419457006020027881 \\
& + 467.918626785158952385003318864 \; \varepsilon \\
& - 1988.539793801800446261028948 92 \; \varepsilon^{2} \\
& - 4066.050049592916043999363931 30 \; \varepsilon^{3} \\
& - 84210.538667468960167279985288 4 \; \varepsilon^{4}
\end{aligned}
$$

Results: few 4-loop masters

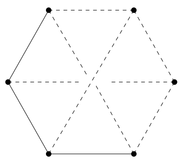


$$= +0.1666666666666666666666666666667\,\varepsilon^{-4}$$
$$+1.50000000000000000000000000000\,\varepsilon^{-3}$$
$$+9.81600647386253928819067413630\,\varepsilon^{-2}$$
$$+46.2790780523770796198432389873\,\varepsilon^{-1}$$
$$+192.143949985189792783133357259$$
$$+895.583232707441960151372330064\,\varepsilon$$
$$+3015.50609158685899428979534043\,\varepsilon^{2}$$
$$+15431.7141565054206165629974866\,\varepsilon^{3}$$
$$+45950.3851577110455999824698200\,\varepsilon^{4}$$

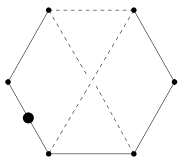


$$= +5.18463877571684963165682743229\,\varepsilon^{-1}$$
$$-19.0793750927079960314257405876$$
$$+141.252248186107747164092632797\,\varepsilon$$
$$-605.029621201870258904662671455\,\varepsilon^{2}$$
$$+2946.48740435117339409606564510\,\varepsilon^{3}$$

Results: few 4-loop masters



$$
\begin{aligned}
= \quad & -6.72847056008568105547188977521 \\
& -26.0876465999666155389659770717\,\varepsilon \\
& -214.647717912411362028052727052\,\varepsilon^2 \\
& -613.715203096626075654874908838\,\varepsilon^3
\end{aligned}
$$



$$
\begin{aligned}
= \quad & +0.473611472272364450 \\
& +1.09585342206826990\,\varepsilon \\
& +5.37764333252884269\,\varepsilon^2 \\
& +8.82896457590640998\,\varepsilon^3
\end{aligned}
$$

Results: On-shell non-singlet $\Delta\rho^{(4)}$

- On-shell 4-loop non-singlet result for ρ:

$$\Delta\rho^{OS,4L}(\text{non-singlet}) = 3X_t\Bigg\{\left(\frac{\alpha_s}{\pi}\right)^3\Bigg[\,C_F^3\Big(-0.7845479837\Big)$$

$$+C_F^2\,C_A\Big(17.20096563-5.918835584\,L_t\Big)$$

$$+C_F^2\,T_F\Big(-0.4393186129+1.616070332\,L_t+n_l\Big(-8.740003239+1.616070332\,L_t\Big)\Big)$$

$$+C_F\,C_A^2\Big(-30.95679757+13.0488891\,L_t-1.802340431\,L_t^2\Big)$$

$$+C_F\,C_A\,T_F\Big(-0.5400590182-5.355886515\,L_t+1.310793041\,L_t^2$$

$$+n_l\Big(24.8274162-9.998375300\,L_t+1.310793041\,L_t^2\Big)\Big)$$

$$+C_F\,T_F^2\Big(0.3035659457+0.09803506636\,L_t-0.2383260074\,L_t^2$$

$$+n_l\Big(0.7160711769+1.884247873\,L_t-0.4766520149\,L_t^2\Big)$$

$$+n_l^2\Big(-3.448039206+1.786212806\,L_t-0.2383260074\,L_t^2\Big)\Big)\Bigg]\Bigg\}$$

Results: On-shell and $\overline{MS}$ $\Delta\rho$

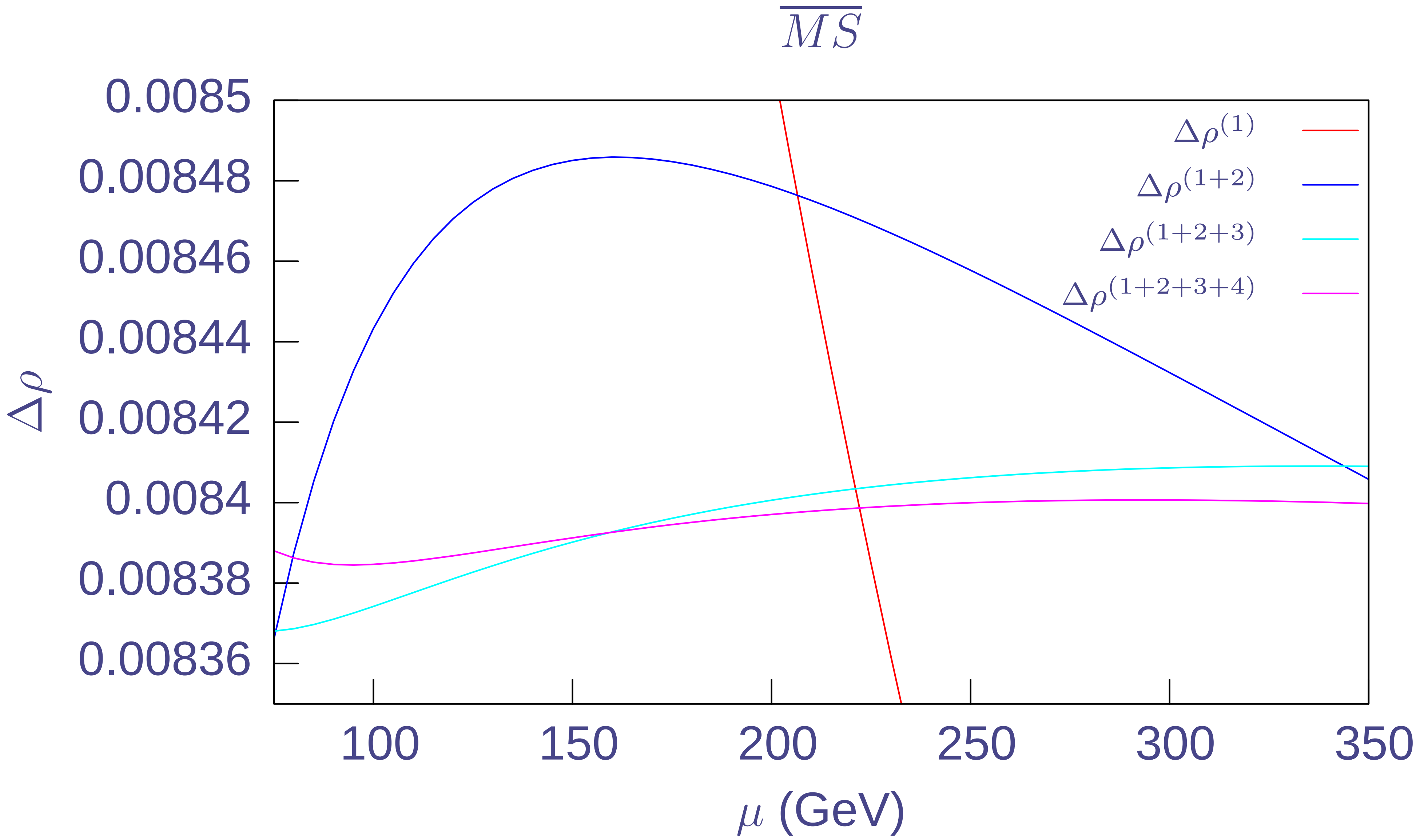
$$X_t = \frac{G_F M_t^2}{8\sqrt{2}\pi^2}, \qquad L_t \equiv \log\left(\frac{M_t^2}{\mu^2}\right),$$

- $\overline{MS}$ and On-shell numerical results

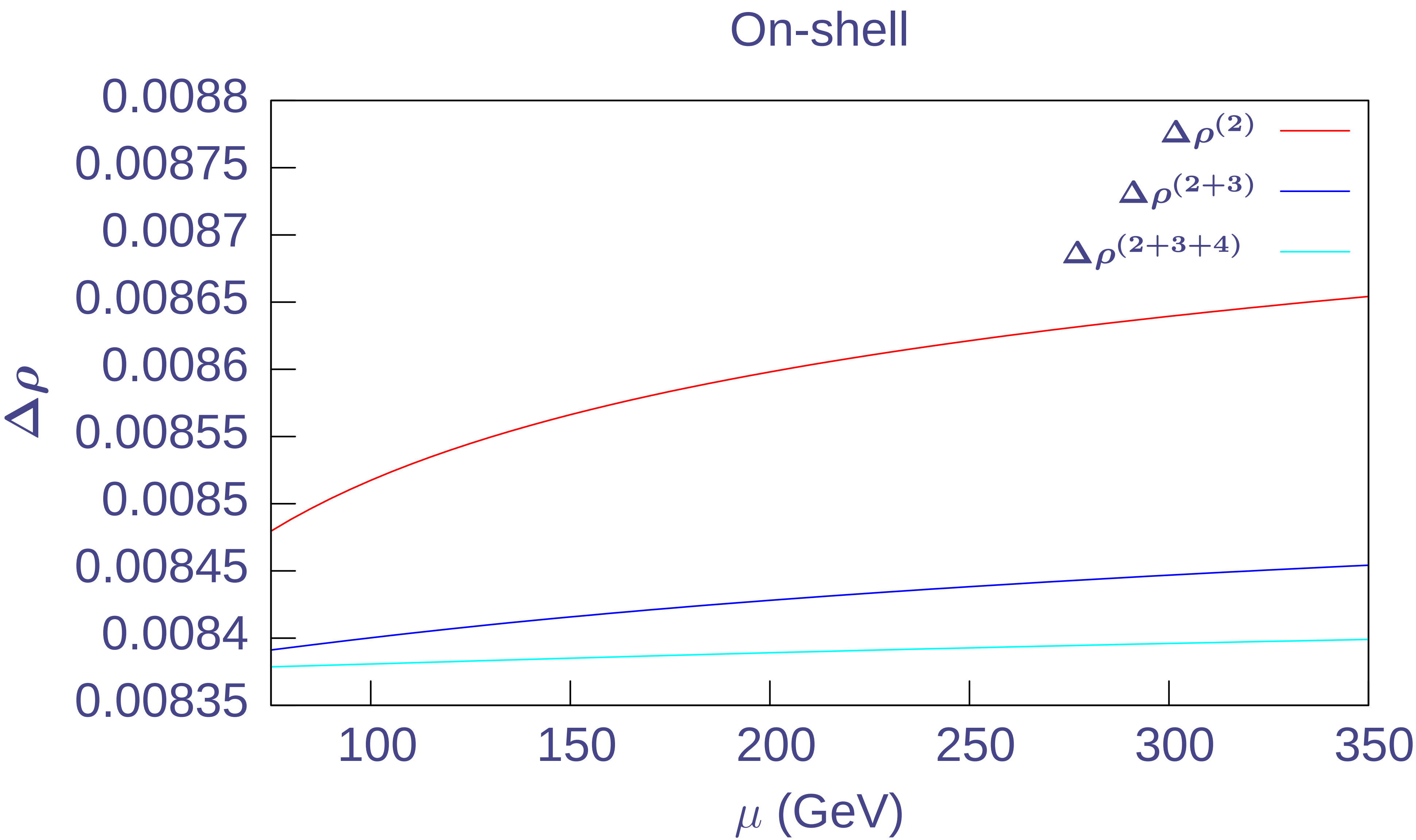
$$\Delta\rho^{\overline{\rm MS}} = 3x_t\left[1 - 0.19325\frac{\alpha_s}{\pi} + (\underbrace{-4.2072}_{\textbf{singlet}} + \underbrace{0.23764)}_{\textbf{non-singlet}}\left(\frac{\alpha_s}{\pi}\right)^2 + (-3.2866 + 1.6067)\left(\frac{\alpha_s}{\pi}\right)^3\right],$$

$$\Delta\rho^{\rm OS} = 3X_t\left[1 - 2.8599\frac{\alpha_s}{\pi} + (\underbrace{-4.2072}_{\textbf{singlet}} \underbrace{-10.387}_{\textbf{non-singlet}})\left(\frac{\alpha_s}{\pi}\right)^2 + (7.9326 - 101.0827)\left(\frac{\alpha_s}{\pi}\right)^3\right],$$

Results: scale dependence of $\Delta\rho^{(\overline{MS})}$



Results: scale dependence of $\Delta\rho^{(\text{On-shell})}$



Conclusions

- New set of high precision numerical values of 4-loop single scale tadpoles is provided
  - important building block in asymptotic expansions
  - depth of expansion is enough for a 5-loop calculation
- The non-singlet contribution completes the 4-loop QCD correction to ρ
  - in the $\overline{MS}$-scheme the singlet contribution is just twice larger than the non-singlet piece
  - in the On-shell scheme the singlet part is 10 times smaller than the non-singlet contribution
- Using $\Delta\rho^{(4)}_{on-shell} = -3.5 \times 10^{-5}$, the shifts to M_W and $\sin^2\theta^{\text{lept}}_{\text{eff}}$ are $\Delta^{(4)} M_W = -2MeV$, $\Delta^{(4)} \sin^2\theta^{\text{lept}}_{\text{eff}} = 1.15 \times 10^{-5}$
  both are below the anticipated precision of future experiments
- Inclusion of the 4L correction to ρ improves the stability of the prediction